MODERN GEOMETRY

BOOKS I—IV

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MODERN GEOMETRY

(MATRICULATION GEOMETRY)

BOOKS I—IV

BY

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REVISED EDITION

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PREFACE

MODERN Geometry, Part I (Matriculation Geometry) comprises Books I to IV, which form the Matriculation Course, Part II containing Book V (Solid Geometry) only. Each Book has been treated on the lines of the syllabus prescribed under the new regulations of the Calcutta University, and I have spared no pains to keep a steady eye on simplicity and clearness at every stage of the students' progress. Books I, II, III, with the exception of Sections III and IV of Book III, may be generally taken to include the Compulsory Course for the Matriculation Examination, whilst Books I to IV may be taken to include the Optional (or Additional) Course.

The order in which Propositions are numbered is not the same in every treatise on modern Geometry. Hence, in all answers meant for the Examiner, the student, while giving a reference, should *never* quote the number of the Proposition referred to, but simply point out its application with clearness and precision. For instance, in proving Theorem 14, Book III, Theorem 10 of the same Book has to be referred to, but all that the student need do is to write out the steps, as given in this treatise, taking care to *omit* "(Th. 10)" at the end of " $\therefore$ the $\angle ADC$ = half the $\angle AOC$".

Model Papers have been inserted at the end of the treatise. The student is strongly recommended to attempt to answer each Paper within the time allotted, and measure his progress by the amount of success attained in such endeavours. It is also advisable that each of these Papers should be answered oftener than once, for a gradually increasing speed and facility may be acquired only by frequent trials.

Any suggestions for the improvement of the work will be thankfully received.

DACCA: *January, 1911.* K. P. BASU

PREFACE TO THE REVISED EDITION

IN this edition, the book has been revised according to the syllabus prescribed by the Government of Bengal (Ministry of Education) for secondary schools.

The sequence of propositions has, for this purpose, been changed so far as it has been found necessary in following the syllabus, and care has always been taken to present the subject-matter in a simple and attractive form, and in a logical order.

The subject has been divided into five sections, the first one dealing with the "lines and angles in rectilineal figures," the second with the "areas of rectilineal figures", the third with the properties of circles", the fourth with the "areas of rectilineal figures connected to circles and the geometrical theorems corresponding to algebraical identities", and the fifth with the "ratio and proportion". As the last one (*viz.* 'ratio and proportion') is prescribed, exclusively for Additional Course, by the Calcutta University, it has been written in a separate book, the present volume containing Books I—IV.

Numerous exercises (both theoretical and practical) have been added, at intervals, to facilitate a thorough grasp of the subject.

Every beginner in Geometry should, first of all, be familiar with the Geometrical instruments and should learn to use them in drawing and measuring geometrical figures without entering into the comparatively abstract theoretical proofs thereof; and because that part of the work is included in the syllabus of Class VI, it has been left out of the present work, which is meant for Classes VII to X.

It is hoped that this edition will be found more useful and attractive than its predecessors.

N. N. SEN

CALCUTTA: *March, 1933.* M. DASGUPTA

Revised syllabuses in Geometry prescribed by the Government of Bengal (Ministry of Education) for Secondary Schools in Bengal.
[Calcutta Gazette November, 13, 1928.]

COMPULSORY COURSE

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CLASS VII

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CHAPTER I

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CHAPTER I

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MODERN GEOMETRY

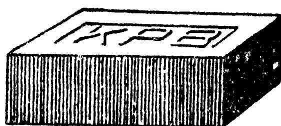
BOOK I

CHAPTER I

FUNDAMENTAL IDEAS AND DEFINITIONS

1. Geometry deals with the position, shape and size of the *space* occupied by bodies.

2. The *space* occupied by any body is called a **solid.**



BRICK



BALL

Consider the portion of space cut off by a brick. Clearly, it has a certain length, a certain breadth and a certain thickness ; likewise, any other solid also has length, breadth and thickness for even if it be not of the form of a brick, it may be easily conceived to be mostly made up of parts, large and small, which are all brick-shaped.

Thus, **a solid has length, breadth and thickness.**

SURFACE, LINE AND POINT

3. That which has *length* and *breadth* only, but *no thickness*, is called a **surface.**

Obs. A solid is bounded by a *surface* or *surfaces.*

For instance, consider a brick-shaped piece of wood. It may be said to have six boundaries, every one of which separates the space occupied by the wood from the rest of space. Each boundary evidently possesses length and breadth, but it has *no thickness* ; for,

if it had any, it would undoubtedly include a slice of the wood or a thin layer of the surrounding air, or both, which it *does not*. Similarly, the boundary of a ball-shaped body has *no thickness*, because the boundary forms a part of neither the body nor that of the surrounding air; but it *has* length and breadth, in the sense that it may be supposed to be divided into small portions, each of which is similar in shape to one or other of the boundaries of a brick-shaped body.

The boundary between two portions of the same solid is also a *surface*.

NOTE. It is evident therefore that a surface occupies *no space*.

4. That which has *length* only, but *neither breadth nor thickness*, is called a **line.**



A line is traced out by moving the sharp end of a pencil on a sheet of paper. But such a trace also has some breadth and some thickness, however small these may be, and will not, therefore, be a line according to our definition. The finer therefore the mark is, the more nearly does it resemble a line.

Example. The boundary between two portions of a surface is a *line*.

For, such a boundary can have length only but neither breadth nor thickness. It cannot have thickness, as it lies on the surface; nor can it have breadth, as in that case it would form a portion of the surface and would not be its boundary.

Likewise, any given surface may be seen to be bounded by a line or lines.

Also, surfaces meet in lines. Thus, the edges of a brick or a box are lines.

5. That which has position only, but *neither length nor breadth, nor thickness*, is called a **point.**

Thus, a point has *position* only but no size, either as to length, breadth or thickness.

The picture of a point is a dot (.) ; but it has still some size and is therefore *not* a point according to our definition. The smaller, however, the dot is, the more nearly does it *resemble* a point.

Example. The boundary between two parts of a given line is a point; because, if this boundary had any length, the two parts would be two distinct lines instead of being parts of the same given line.

It is also clear that the extermities of a line are points.

6. Thus, *points*, *lines*, *surfaces* and *solids* are related to each other as follows :

(i) Boundaries of solids are surfaces.

(ii) Boundaries of surfaces are lines; two surfaces intersect in lines.

(iii) Boundaries (*i.e.* extremities) of lines are points: two lines intersect in points.

Also, if a point moves from one position to another, a line is generated, of which the two positions of the point are the extremities. For example, (see § 4) a line is traced by moving a sharp pencil point on a sheet of paper. Similarly, a *moving line* traces a *surface* and a *moving surface* traces a solid.

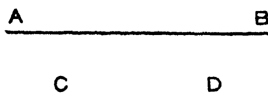
7. Of the three things—*viz.* length, breadth and thickness, each is called a **dimension.** That which possesses one or more of these dimensions is called a **magnitude.**

Hence, (*i*) *a* ***solid*** *is a magnitude of* ***three dimensions ;*** (*ii*) *a* ***surface*** *is a magnitude of* ***two dimensions ;*** (*iii*) *a* ***line*** *is a magnitude of* ***one dimension ;*** (*iv*) *a* ***point*** *is* ***not a magnitude,*** *and it has* ***no dimension.***

Straight Lines

8. If two lines be such that in whatever manner they coincide in two points they invariably become indistinguishable from one another, each of them is called a **straight line** or a **right line.**

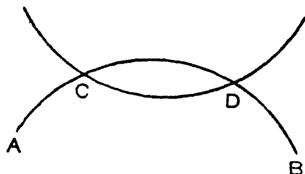
Example. Let AB, CD be two lines, as in the following diagram.



Suppose the line CD to be taken up and placed in such a way that the points C and D fall upon the line AB; on whatever points on the line AB and in whatever manner, the points C and D may fall, if in every case the two lines become indistinguishable one from the other, then *each* of the lines AB and CD is a straight line.

In the following diagram the two lines AB and CD are quite distinguishable from one another, even though they coincide in two points.

In this case, therefore, the lines AB and CD are *not* straight lines.



Properties of Straight Lines

From the above definition the following conclusions are evident :—

(1) *Two straight lines cannot have more than one point common.*

For, if they had two points common, they would completely become one and the same straight line. This is otherwise expressed by saying that *two straight lines cannot intersect each other in more points than one.*

(2) *Two straight lines cannot have a common segment* (a part of a line is called its *segment*).

For, if they coincide partly, they certainly coincide in two points, and therefore they must be indistinguishable throughout.

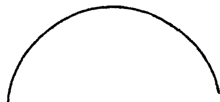
(3) Through one given point an unlimited number of straight lines may pass; but *only one straight line can pass through two given points.*

9. A straight line whose extremities are given is called a **finite** or **limited** straight line, whilst one which may be supposed to be extended to any length either-way, is called an **unlimited** straight line.

10. *The* ***mid-point*** *of a given straight line* is the point which divides it into two equal parts.

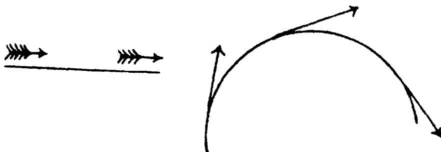
11. One finite straight line is *equal* to another when either of them may be made to coincide with the other.

12. A line which is neither straight, nor made up of straight lines, is called a **curved line,** or, more simply, a **curve.**



The accompanying diagram is the picture of a curved line.

When a point moves along a straight line its direction of motion remains unchanged, whilst if a point is moving along a curve its direction of motion constantly changes. In the accompanying pictures the arrow-heads shew the directions of motion.



13. The **distance** between two points is the length of the straight line joining them.

PLANE

14. If a surface be such that the straight line passing through *any* two points in it lies wholly in that surface, it is called a **plane surface,** or, more simply, a **plane.**

If the surface be unlimited all round, the straight line passing through the two points should also be considered as unlimited in both the directions. This unlimited straight line then, for all position of the points, will be *completely* in contact with the surface, *if it is a plane.*

If two points be taken on the surface of a ball-shaped body, the straight line passing through them does not at all lie in that surface. Clearly therefore such a surface *is not* a plane surface.

N. B. Henceforth, unless the contrary is stated, *all points and lines shall be supposed to be existing in one and the same plane.*

ANGLES

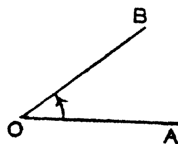
15. Two straight lines drawn from a point are said to form an **angle** at that point.

The straight lines are called the **arms** of the angle and the point at which they meet is called the **vertex** of the angle.

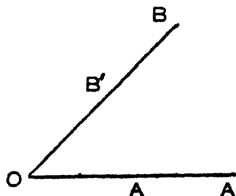
The angle contained by the straight lines OA *and* OB is briefly expressed as "*the angle* AOB", or "*the angle* BOA", the letter O being always put in the middle.

The size of the angle may be thus explained :

If two straight lines OA and OB be drawn from O, the straight line OB may be supposed to have originally coincided in direction with OA, and then to have moved from that position into its present one by *rotation* about the point O ; it is this *amount of rotation* which is called the **angle** contained by the straight lines OA and OB.



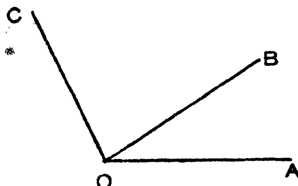
NOTE. Clearly, the size of an angle does not at all depend upon the *lengths* of its arms. Thus, in the accompanying diagram, the angle AOB is the same as the angle A′OB′.



16. Two *angles* are said to be equal when the arms of the one may be made to coincide with those of the other, each to each.

17. The **bisector of an angle** is the straight line which divides the angle into two equal parts.

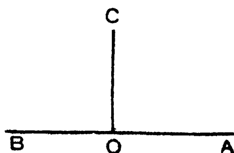
18. Two angles are said to be **adjacent** when they have a common vertex and lie on opposite sides of a common arm.



Thus, the angles AOB and BOC which are on opposite sides of the common arm OB are *adjacent*.

Right Angle

19. If a straight line OC standing upon another straight line AOB make the adjacent angles AOC, COB equal to one another, each of these angles is called a **right angle**; and either of the two straight lines is said to be **perpendicular** to the other, or, at **right angles** to the other.

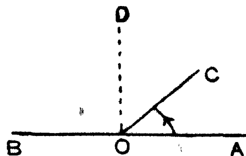


If a right angle be supposed to be divided into 90 equal parts, each part is called a **degree**; a sixtieth part of a degree is called a **minute**, and a sixtieth part of a minute is called a **second**; *i.e.*,

1 right angle = 90 degrees (90°)
1 degree = 60 minutes (60′)
1 minute = 60 seconds (60″).

20. At any given point in a straight line there can be only one perpendicular to it.

For, let AOB be any straight line, and OC another which starting from the position OA rotates about O in the direction of the arrow-head, as shewn in the accompanying diagram.



As OC goes on rotating, it is evident that the angle AOC gradually increases whilst the angle COB gradually decreases. Clearly therefore there is one position of OC, *and one only*, for

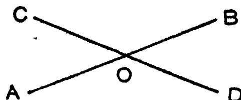
which the angle AOC is equal to the right angle COB; and it is in this position that OC is perpendicular to AB.

NOTE. As OC rotates about the point O from the position OA into the position OB, it turns through two right angles. Hence, when the straight lines OA and OB are in the same straight line but *in opposite directions*, the angle AOB is equal to *two right angles*.

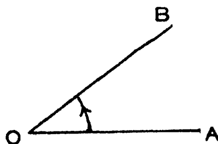
21. When OA and OB are in the same straight line but in opposite directions, the angle AOB is called a **straight angle.** Therefore, *a straight angle* = 2 *right angles* = 180°.

22. When two straight lines cross each other, the two pairs of opposite angles are called **vertically opposite.**

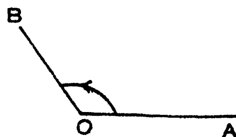
Thus, the angles AOC and BOD are vertically opposite; and so are the angles COB and DOA.



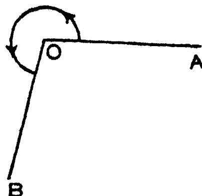
23. An angle which is less than a right angle is called an **acute angle.**



An angle which is greater than one right angle but less than two right angles, is called an **obtuse angle.**



An angle which is greater than two right angles but less than four right angles, is called a **reflex** or **re-entrant angle.**

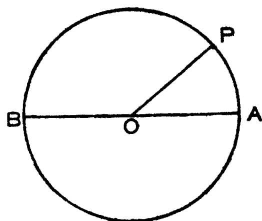


24. Any portion of a plane surface bounded by one or more lines, is called a **plane figure.**

The amount of surface enclosed by the boundaries of a plane figure is called its **area.**

NOTE. In a more extended sense, any combination of points and lines is called a *figure.* The picture of a figure is called a *diagram.* Real points and lines can never be constructed ; so whatever is drawn on paper is only a *picture* of the real thing. From this it is clear that when a figure is drawn on paper it is the diagram that is really drawn, and the finer the diagram the more nearly does it represent the figure intended.

25. A **circle** is a plane figure bounded by one line which is such that all straight lines drawn to it from a certain point within the figure are equal to one another.



The bounding line of a circle is called its **circumference.**

The point within a circle from which all straight lines drawn to the circumference are equal, is called the **centre** of the circle.

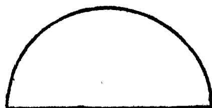
Any straight line drawn from the centre of a circle to the circumference is called a **radius** of the circle.

Hence, all radii of the same circle are equal to one another.

Any straight line drawn through the centre of a circle and terminated both ways by the circumference is called a **diameter** of the circle.

Thus, in the above figure, O is the *centre,* AOB is a *diameter,* each of OP, OA, OB is a *radius.* Evidently, a diameter $=$ $2 \times$ radius.

26. A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference which it cuts off.



Every diameter divides a circle into two semi-circles.

CHAPTER II

EXPLANATIONS OF TERMS : POSTULATES AND AXIOMS

27. Plane geometry deals with the properties of lines, angles and figures drawn on a plane surface.

28. A complete discussion of a geometrical truth, or of a geometrical construction is called a **Proposition.**

29. Propositions are of two kinds :—*Theorems* and *Problems*.

30. A proposition in which a geometrical truth is stated and proved is called a **Theorem.**

31. A proposition in which a geometrical construction is proposed and effected is called a **Problem.**

32. A proposition consists of *four* parts :—(i) the *General Enunciation* ; (ii) the *Particular Enunciation* ; (iii) the *Construction* and (iv) the *Proof*.

(i) The **General Enunciation** is the statement in general terms of what is to be proved, or of what is to be done.

It consists of two parts :—

(1) What is *assumed to be true*—called **Hypothesis,**

and (2) what is *required to be proved*—called **Conclusion.**

(ii) The **Particular Enunciation** is the statement of what is to be proved, or of what is to be done, *with special reference to a particular diagram.*

(iii) The **Construction** directs the drawing of such straight lines and circles as may be required to prove the truth of a theorem, or to accomplish the object of a problem.

(iv) The **Proof** shows that the truth of a theorem has been established or the object of a problem has been attained.

NOTE. In the case of a theorem, the construction may be regarded as forming a part of the Proof.

33. A geometrical truth which can be easily deduced from a theorem is called a **corollary** to the theorem.

POSTULATES AND HYPOTHETICAL CONSTRUCTIONS

34. For the purpose of effecting geometrical constructions certain constructions that may be easily admitted as possible are made the foundation of others which are not so obvious. These simple and easily admissible constructions are called **postulates.** They are three in number :—

(1) *A straight line may be drawn from any one point to any other point.*

(2) *A finite (or terminated) straight line may be produced to any length along that straight line.*

(3) *A circle may be drawn with any point as centre and with a radius equal to any finite straight line.*

NOTE. The first two constructions may be practically carried out with the help of a flat ruler, whilst the third requires the use of a pair of compasses which may be so adjusted as to transfer a distance from one position to another.

35. Some constructions which are not so obvious as the postulates, and which, in fact, may have to be performed with the help of one or more of them, may as well be *taken for granted* for the purpose of establishing a geometrical truth. A construction, which is thus assumed before it is shewn *how* it can be performed, is called a **hypothetical construction.** It is clear therefore that a construction which is hypothetical in one place is not necessarily so in another. The following constructions will be assumed before they are formally proved :—

(1) *A perpendicular may be drawn to a given straight line from any given point in that straight line.*

(2) *A straight line may be drawn to bisect any given angle.*

(3) *A finite straight line may be bisected at a point.*

N.B. For another such hypothetical construction see Art. 60.

Axioms

36. An **axiom** is a self-evident truth. The following axioms are useful in geometrical proofs.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the sums are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the sums are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are doubles of the same thing, or of equal things, are equal to one another.

7. Things which are halves of the same thing, or of equal things, are equal to one another.

8. Magnitudes (lines, angles or figures) which can be made to coincide with one another are equal.

NOTE. The process of taking up one magnitude from its position and placing it upon another is called **superposition**; and the former magnitude is said to be **applied** to the latter.

9. The whole is greater than its part.

N.B. It is also obvious that the whole is equal to the sum of its parts.

10. Two straight lines which coincide in their extremities coincide completely.

N.B. This is equivalent to saying that "two straight lines cannot enclose a surface".

This follows from Art. 8.

11. All right angles are equal.

This follows from Art. 19.

N.B. For Axiom 12, see Art. 59.

EXERCISE 1

1. Define a *point*, a *line*, a *surface* and a *solid*. How are they related to one another ? What are their dimensions ?

2. Is a very small ink-spot on paper a point ? If not, why ?

3. Is a straight line drawn on paper really a *geometrical straight line* ? If not, why ?

4. If a finite straight line be supposed to be divided into a very large number of equal parts, has any of these parts any *dimension* ? If so, what ?

5. What kind of magnitude is the *boundary* between the calm water of a tank and the air above ? and why ?

6. Define a straight line. Show that two straight lines cannot meet in more points than one.

7. Define a *plane*. On a plane surface how many straight lines can be drawn through (i) one point ; (ii) through two points ?

8. If two plane surfaces have two points A and B in common, prove that the straight line drawn from A to B is also common to the two planes.

9. If the extremities of one finite straight line fall upon those of another, two lines are equal. Why ?

10. If the paper on which a straight line AB is drawn be so folded that the point A comes upon the point B, and if C be the point on AB through which the *crease* passes, then C is the middle point of AB. Why ?

11. Define an angle. Does the size of an angle depend on the length of its arms ?

When does one angle coincide with another ?

12. Define a *right angle*, a *reflex angle* and *adjacent angles*.

If the paper on which a straight line AB is drawn be so folded that one portion of the line falls upon another, and if O be the point on AB through which the crease passes, and if C be any other point on the crease, prove that OC is perpendicular to AB.

13. Define a straight angle. Show that a straight angle is equal to 180°.

14. If a straight line starting from the position OA continually rotates about O in the same direction until it comes back to its old position, it turns through four right angles. How ?

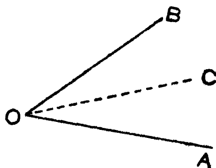
15. AB and CD are two finite straight lines of which AB is greater than CD. Show how the third postulate can be applied in cutting off from AB a part equal to CD.

16. Show how the postulates can be applied in drawing from a given point A a straight line equal to a given straight line BC.

17. Two finite straight lines AB and CD are equal to one another. If E be a point on AB and F be a point on CD such that AE is equal to CF, then EB is also equal to FD. Quote the axiom by which this is proved.

18. If one angle ABC be applied to another angle PQR so that B coincides with Q and BC falls upon QR, it is found that BA occupies an intermediate position between QR and QP. Quote the axiom by which it is proved that the angle ABC is less than the angle PQR.

19. AOB is an angle. If the paper be folded so that OA falls on OB and if the straight line OC marks the crease, then OC is the bisector of the angle AOB. How ?



20. What are the four parts of a proposition ? Explain the object of each part.

21. Define *Hypothesis*, *Conclusion*, *Corollary*, *Postulates* and *Axioms*.

22. Distinguish between a *theorem* and a *problem*.

SYMBOLS AND ABBREVIATIONS USED IN GEOMETRY

37. The following symbols and abbreviations may be used in writing out propositions :—

$\therefore$	*for*	therefore.
$\because$	,,	because.
$=$	,,	equal to ; is, or are, equal to.
$\angle$	,,	angle.
Rt. $\angle$	,,	right angle.
$\triangle$	,,	triangle.
Par' (or $\parallel$)	,,	parallel.
Perp. or, $\perp$	,,	perpendicular.
$\odot$	,,	circle.
$\bigcirc''$	,,	circumference
$>$	,,	greater than ; is, or are, greater than.
$<$	,,	less than ; is, or are, less than.
Pt.	,,	Point.
Str.	,,	Straight line.
Par^{m}.	,,	Parallelogram.
Sq.	,,	Square.
Rect.	,,	Rectangle.

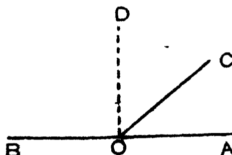
Rectil	*for*	Rectilineal.
Diff.	,,	Different.
Opp.	,,	Opposite.
Adj.	,,	Adjacent.
Diag.	,,	Diagonal.
Def.	,,	Definition.
Post.	,,	Postulate.
Ax.	,,	Axiom.
Cons.	,,	Construction.
Hyp.	,,	Hypothesis.
Equilat.	,,	Equilateral.
Isos.	,,	Isosceles.
Quadl.	,,	Quadrilateral.
Reqd.	,,	Required.
Int.	,,	Interior.
Ext.	,,	Exterior.
Prop.	,,	Proposition.
Th.	,,	Theorem.
Cor.	,,	Corollary.
Join AB	,,	Draw the straight line from A to B.

CHAPTER III

THEOREMS ON ANGLES AT A POINT

THEOREM 1.

If a straight line stands on another straight line, the sum of the two adjacent angles, so formed, is equal to two right angles.



Let the straight line OC stand on the straight line AB and form the adjacent angles AOC, COB.

It is required to prove that

$$\angle AOC + \angle COB = 2 \text{ right angles.}$$

Construction. Suppose OD is drawn perpendicular to the straight line AB.

Proof. $\angle AOC + \angle COB = \angle AOC + \angle COD + \angle DOB.$

Also, $\angle AOD + \angle DOB = \angle AOC + \angle COD + \angle DOB,$

$\therefore \quad \angle AOC + \angle COB = \angle AOD + \angle DOB.$

But, by construction, $\angle AOD$ and $\angle DOB$ are right angles.

$\therefore \quad \angle AOC + \angle COB = 2$ right angles. Q. E. D.*

Another Proof.

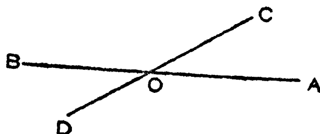
$\angle AOC + \angle COB =$ the *straight* angle AOB

$=$ two right angles. (*§ 21*)

Q. E. D.

*The letters Q. E. D. which are found at the end of a theorem stand for the Latin words **Quod Erat Demonstrandum,** meaning "*which was to be proved*".

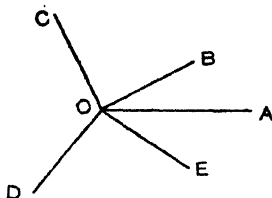
COROLLARY 1. *If two straight lines cut one another, the four angles thus formed are together equal to four right angles.*



Thus, $\angle AOC + \angle COB + \angle BOD + \angle DOA = 4$ right angles.

COR. 2. *If any number of straight lines diverge from a given point, the sum of the consecutive angles so formed is equal to four right angles.*

In this case, $\angle AOB + \angle BOC + \angle COD + \angle DOE + \angle EOA = 4$ right angles.



For, if any one of the lines be produced, the sum of the angles on *each* side of this line = two rt. $\angle$s; and $\therefore$ $\angle AOB + \angle BOC + \angle COD + \angle DOE + \angle EOA$ = the sum of the $\angle$s on both sides = four right angles.

38. When two angles are together equal to two right angles, each of them is called the **supplement** of the other, and the two angles are said to be **supplementary** to each other.

Thus, in the diagram of Theorem 1, each of the angles AOC and COB is the *supplement* of the other. Likewise, 50° is supplementary to 130°, because $50° + 130° = 180°$.

39. When two angles are together equal to one right angle, each of them is called the **complement** of the other, and the two angles are said to be **complementary** to each other.

Thus, in the diagram of Theorem 1, each of the angles AOC and COD is the *complement* of the other. In like manner, 30° is complementary to 60°.

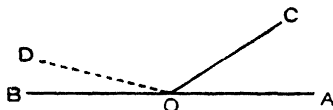
COR. 3. (i) *Supplements of the same angle or of equal angles are equal.*

(ii) *Complements of the same angle or of equal angles are equal.*

Ex. Show that supplement of an acute angle is obtuse and that of an obtuse angle is acute.

THEOREM 2.

If two adjacent angles are supplementary, their exterior arms are in the same straight line.



Let the adjacent angles AOC and COB be supplementary.

It is required to prove that OB *is in the same straight line with* AO.

Proof. Now, OB may be either in the *same str. line* with AO, or *not* so.

Suppose OB is *not* in the same str. line with AO, and that OD, which is different from OB, is the produced part of AO.

Now, since OA and OD are in the same str. line,

$\therefore \quad \angle AOC + \angle COD =$ two right angles.

But, by hypothesis, $\angle AOC + \angle COB =$ two right angles,

$\therefore \quad \angle AOC + \angle COD = \angle AOC + \angle COB.$

From these equals, take away the common $\angle AOC$;

$\therefore \quad \angle COD = \angle COB;$

i.e., a part is equal to the whole, which is *impossible*.

Thus, it is *absurd* to suppose that OB is *not* in the same straight line with AO,

$\therefore$ OB is in the same str. line with AO.

Q. E. D.

If two theorems be so related that *the hypothesis of each is the conclusion of the other*, then either of them is said to be the **converse** of the other.

40. Now, in Theorem 1, the *hypothesis* is that *the arms*, OA *and* OB of the adjacent angles AOC and COB *are in the same straight line* and the *conclusion* is that *the angles* AOC *and* COB *are supplementary.*

But, in Theorem 2, the *hypothesis* is that *the angles* AOC and COB *are supplementary*; and the *conclusion* is that the *arms,* OA and OB, *are in the same straight line.* Clearly therefore the hypothesis and conclusion of Theorem 2 are respectively the conclusion and hypothesis of Theorem 1. Hence, Theorem 2 is the converse of Theorem 1.

41. The method of proof used in this proposition is called the **Indirect Method** or **reductio ad absurdum.**

It is quite clear that the straight line OB must *either be* in the same straight line with AO, *or not*; hence when the second supposition is proved to be *absurd,* the inevitable conclusion is that the first is correct. Thus, the **indirect method** consists in proving the truth of a theorem *by proving the absurdity of the supposition that the theorem is not true.* It is generally employed in proving the converse of an established theorem.

EXERCISE 2

1. Write down the supplements of (i) 30°; (ii) 45°; (iii) 67°; (iv) 120°; (v) $\frac{2}{3}$ of a rt. angle; (vi) 115° 23′; (vii) 25° 50′ 35″; (viii) 135° 31′ 47″; (ix) $\frac{2}{3}$ of a str. angle.

2. Write down the complements of (i) 30°; (ii) 45°; (iii) 60°; (iv) $\frac{1}{3}$ of a rt. angle; (v) $\frac{2}{3}$ of a rt. angle; (vi) $\frac{1}{10}$ of a str. angle; (vii) 35° 50′; (viii) 61° 32′ 15″; (ix) 81° 17′ 47″.

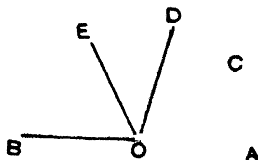
3. If, in the figure of Theorem 1, ∠AOC be (i) 45°, (ii) 60°, or (iii) 85° 35′, find ∠BOC in each case.

4. If, in the figure of Theorem 1, ∠COD is (i) 30°, (ii) 15°, or (iii) 45°, find ∠AOC in each case.

5. If, in the figure of Theorem 1, ∠BOC is (i) 120°, (ii) 135°, or (iii) 150°, find ∠COD in each case.

6. If each of the straight lines OC, OD, OE meet the straight line AB, on the same side of it, at O, find the sum of the angles AOC, COD, DOE and EOB.

If ∠AOC=25°, ∠DOE=35° and ∠EOB=30°, find ∠COD.

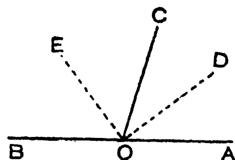


7. If two straight lines cut one another and if one of the four angles so formed be a right angle prove that each of the other three angles is also a right angle.

8. Prove that the internal and external bisectors of an angle are at right angles to each other.

NOTE. The bisector of an angle and that of its supplement obtained by producing one of its arms are called the **internal** and **external bisectors** of the angle.

Thus, in the figure, OD and OE are the internal and external bisectors of the ∠AOC.



9. Show that in the above diagram (i) ∠AOD and ∠BOE are complementary; (ii) ∠AOE and ∠COE are supplementary; (iii) ∠BOD and ∠COD are supplementary.

10. In the diagram of COR. 2 (Th. 1), if ∠AOB=30°, ∠BOC =60°, ∠DOE=135° and ∠EOA=75°, find ∠COD.

11. If four straight lines OA, OB, OC, OD diverge in order, from a point O, so that each of the ∠s AOB, BOC, COD, DOA is a right angle, prove that AO, OC are in the same straight line, and so are BO, OD.

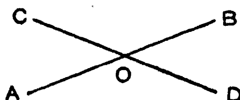
12. Prove that the bisectors of the four angles made by two intersecting straight lines form two perpendicular straight lines.

13. If two straight lines OC, OD diverge on opposite sides of the straight line AOB, so that the ∠AOC=the ∠BOD, prove that OD is in the same straight line with CO.

14. If four straight lines OA, OB, OC, OD diverge from a point O, so that the ∠AOB=the ∠COD and the ∠BOC=the ∠DOA, prove that AO, OC are in the same straight line, and so are BO, OD.

THEOREM 3.

If two straight lines intersect, the vertically opposite angles are equal.



Let the straight lines AB, CD intersect at the point O.

It is required to prove that

(1) $\angle AOC = \angle BOD$

and (2) $\angle COB = \angle DOA$.

Proof. Because OC stands on the straight line AB

$\therefore \angle AOC + \angle COB =$ two rt. angles;

and because OB stands on the straight line CD

$\therefore \angle COB + \angle BOD =$ two rt. angles.

$\therefore \angle AOC + \angle COB = \angle COB + \angle BOD$.

From these equals, take away the common $\angle COB$.

$\therefore \angle AOC = \angle BOD$.

Similarly, $\angle COB = \angle DOA$. Q. E. D.

EXERCISE 3

(*In the diagram of Theorem 3.*)

1. If $\angle AOC = 30°$, find each of the remaining angles by calculation.

2. If $\angle COB = 120°$, write down the values of each of the remaining angles.

3. If $\angle AOC + \angle BOD = 90°$, find all the angles by calculation.

4. If $\angle AOC + \angle BOC + \angle AOD = 270°$, find $\angle BOC$ and $\angle BOD$.

5. Two straight lines AB, CD cut one another at O. Prove that the bisector of the $\angle AOC$, when produced through O, also bisects the $\angle BOD$.

6. Prove that the bisectors of two vertically opposite angles are in one and the same straight line.

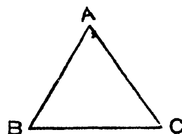
On Triangles

42. A plane figure which is bounded by straight lines only is called a **plane rectilineal figure.**

The bounding straight lines are called the **sides** of the figure.

The sum of the sides of a plane rectilineal figure is called its **perimeter** ; and the portion of the surface enclosed by the sides is called its **area.**

43. A plane figure, bounded by *three* straight lines is called a **triangle.**

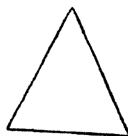


Any angular point of a triangle is called a **vertex** of the triangle, so that a triangle has *three vertices.*

But when a particular side is spoken of as the **base** of the triangle, *it is then the opposite angular point alone that is called the vertex.* Thus, in the above diagram, when the side BC is regarded as the base of the triangle, the angular point A is called the vertex.

In any triangle the straight line drawn from a vertex to the middle point of the opposite side is called a **median.**

44. A triangle whose three sides are equal is called an **equilateral triangle.**

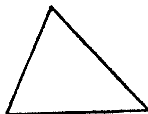


45. A triangle which has two sides equal is called an **isosceles triangle.**

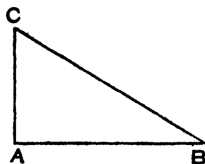


In an isosceles triangle the point of intersection of the equal sides is generally spoken of as its vertex and the opposite side as the base. The angle contained by the equal sides is called the **vertical angle** of the triangle.

46. A triangle, which has three unequal sides is called **a scalene triangle.**



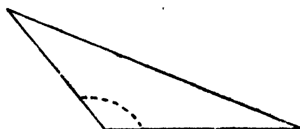
47. A triangle one of whose angles is a right angle, is called a **right-angled triangle.**



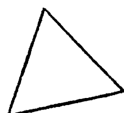
In a right-angled triangle, the side opposite to the right angle is called the **hypotenuse.**

Thus, in the above diagram, the side BC is the hypotenuse of the triangle ABC which is right-angled at A.

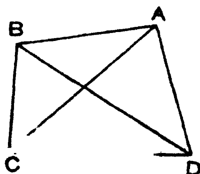
48. A triangle one of whose angles is an obtuse angle, is called an **obtuse-angled triangle.**



49. A triangle of which all the angles are acute is called an **acute-angled triangle.**



50. A plane figure bounded by *four* straight lines is called a **quadrilateral.**

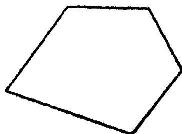


A straight line joining two opposite angular points of a quadrilateral is called a **diagonal** of the quadrilateral.

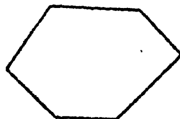
Thus, in the above diagram, the straight lines AC and BD are diagonals.

51. A plane figure bounded by *more* than *four* straight lines is called a **polygon.**

52. A polygon of five sides is called a **pentagon.**



A polygon of six sides is called a **hexagon.**



53. A rectilineal figure is called **equiangular,** when all its angles are equal, and **equilateral,** when all its sides are equal; and **regular,** when it is both equilateral and equiangular.

On the Congruence of Triangles

54. Every triangle has *six parts,* namely the three sides and the three angles. Each triangle has an area also.

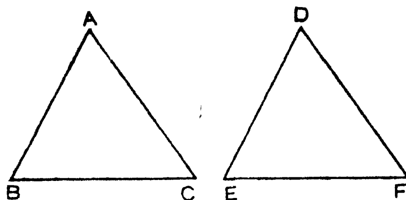
Two triangles are said to be *equal in all respects* when all the six parts of the one are respectively equal to all the six parts of the other.

If a triangle can be so placed on another as to exactly coincide with it, clearly they are equal in all respects and their areas are also equal. Two such triangles are said to be **identically equal** or **congruent.**

In congruent triangles, sides opposite to equal angles are called **corresponding sides** and angles opposite to equal sides are called **corresponding angles.**

THEOREM 4.

If two triangles be such that two sides and the included angle of one are respectively equal to two sides and the included angle of the other, then these two triangles are equal in all respects.



Let ABC, DEF be two triangles such that

AB = DE,

AC = DF,

and the included ∠BAC = the included ∠EDF.

It is required to prove that the △ACB = *the* △DEF, *in all respects.*

Proof. Apply the △ABC to the △DEF, so that A falls on D and AB on DE. Then, because the ∠BAC = the ∠EDF, *Hyp.*

∴ AC must fall upon DF.

Now, since AB falls on DE, and AB = DE,

∴ B falls on E ;

and, since AC falls on DF, and AC = DF,

∴ C falls on F.

∴ BC coincides with EF.

Thus, the sides AB, AC, BC coincide respectively with DE, DF, EF ;

and ∴ the △ABC coincides with the △DEF.

Hence the △ABC = the △DEF, in all respects.

Q. E. D.

NOTE. When we say that the triangles are equal *in all respects*, this certainly includes the three equalities which form the *hypothesis* AB = DE, AC = DF, ∠A = ∠D. Hence, to make a clear

distinction between hypothesis and conclusion, we should say that the *conclusion* is that BC = EF, ∠B = ∠E, ∠C = ∠F, and the area of △ABC = the area of △DEF. That is, of the seven equalities possible, if the above three hold good, the other four *necessarily* follow.

N. B. Notice that in two congruent triangles, *angles opposite to equal sides* (*i. e.*, corresponding angles) *are equal*, as is evident from the above diagram.

EXERCISE 4

1. Prove that the bisector of the vertical angle of an isosceles triangle bisects the base and is also perpendicular to it.

2. AB is a given straight line. Another straight line CD passes through the middle point of AB and is also perpendicular to it. If P be any point on CD prove that PA = PB.

3. Two lines AB, CD bisect each other at O. Prove that the triangles BOD and AOC are equal in all respects ; and the triangles BOC, AOD are equal in all respects.

4. Through D, the middle point of the side BC of the triangle ABC, AD is drawn and produced to E such that DE = AD. Prove that

   (i) the triangles BDE and ADC are congruent ;

   (ii) the triangles CDE and ADB are congruent.

5. The diagonals AB, CD of the quadrilateral ACBD bisect each other at right angles. Prove that the sides of the quadrilateral are equal.

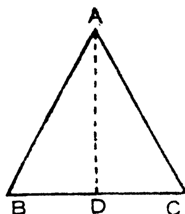
6. In the triangle ABC, the sides AB, AC are bisected at D and E. If DO and EO are drawn perpendiculars to AB, AC respectively, meeting each other at O, prove that OA = OB = OC.

7. In the quadrilateral ABCD, the sides AB, CD are equal and ∠ABC = ∠BCD. Prove that its diagonals are equal.

8. If ABCDE be a regular pentagon, prove that the triangle ACD is isosceles.

THEOREM 5.

If two sides of a triangle are equal, the angles opposite to these two sides are equal.



Let ABC be a Δ, having the side AC = the side AB.

It is required to prove that the $\angle$ ABC = *the* $\angle$ ACB.

Construction. Let AD bisect the $\angle$BAC, and let it meet BC in D.

Proof. In the Δ^s BAD, CAD,

AB = AC

AD is common to both triangles,

and the included $\angle$BAD = the included $\angle$CAD.

$\therefore$ the Δ^s are equal in all respects, *Th. 4.*

so that, the $\angle$ABD = $\angle$ACD

i.e., the $\angle$ABC = $\angle$ACB. Q. E. D.

COR. *If a triangle is equilateral, it is also equiangular.*

EXERCISE 5

1. If the equal sides AB, AC of the isosceles triangle ABC be produced to D and E respectively, prove that $\angle$BCE = $\angle$CBD.

2. The isosceles triangles ABC, DBC stand on the same base BC, show that $\angle$ABD = $\angle$ACD.

3. ABCD is a quadrilateral with all its sides equal. Prove that $\angle$BAD = $\angle$BCD and $\angle$ABC = $\angle$ADC.

4. On the base BC of the isosceles triangle ABC, two pionts P and Q are taken such that BP = CQ. Prove that the triangle APQ is isosceles.

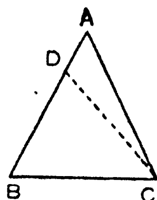
5. Prove that the triangle formed by joining the middle points of the sides of an isosceles triangle is also isosceles.

6. Prove that the triangle formed by joining middle points of the sides of an equilateral triangle is also equilateral.

THEOREM 6.

[CONVERSE OF THEOREM 5]

If two angles of a triangle are equal, the sides opposite to these two angles are equal.



Let ABC be a Δ, having the $\angle$ABC = the $\angle$ACB.

It is required to prove that the side AC = *the side* AB.

Proof. Suppose, AC is not equal to AB. Then one of them must be greater; let AB > AC, and from BA cut off BD = AC.

Join DC.
Then in the Δ^s DBC, ACB,
because DB = AC,
BC is common to both,
and the included $\angle$DBC = the included $\angle$ACB; *Hyp.*
$\therefore$ the ΔDBC = the ΔACB; *Th. 4.*

i.e., a part is equal to the whole, which is *impossible.*

Thus it is *absurd* to suppose that AC is *not* equal to AB.

$\therefore$ AC = AB.

Q. E. D.

COR. *If a triangle is equiangular, it is also equilateral.*

EXERCISE 5A

1. The base BC of a triangle ABC is produced both ways. If the exterior angles at B and C are equal, prove that the triangle is isosceles.

2. Prove that the angles opposite to unequal sides of a triangle are also unequal.

3. Bisectors of the base angles ∠ABC, ∠ACB of an isosceles triangle meet at O. Prove that OB=OC, and AO bisects BC at right angles.

4. In Ex. 1, of Exercise 5, if the bisectors of the angles BCE and CBD meet at O, prove that the triangle BOC is isosceles and that AO bisects BC at right angles.

5. ABCD is a quadrilateral such that the sides AB, AD are equal and the angles ABC and ADC are equal. Prove that the triangle BCD is isosceles.

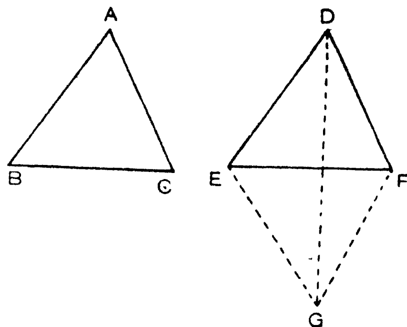
6. The bisectors of the angles ABC, ACB of a triangle ABC meet at O. If BO and CO are equal, prove that the triangle ABC is isosceles.

7. In Ex. 6, if the external bisectors of the angles ABC, ACB meet at O such that BO and CO are equal, prove that the triangle ABC is isosceles.

8. The equal sides AB, AC of an isosceles triangle are produced to D and E respectively, such that BD=CE ; if the bisectors of the angles CBD, BCE meet at O, prove that OD=OE.

THEOREM 7.

If two triangles be such that the three sides of one are respectively equal to the three sides of the other, the two triangles are equal in all respects.



Let ABC and DEF be two Δ^s, having
AB = DE,
AC = DF,
and BC = EF.

It is required to prove that the two Δ^s are equal in all respects.

Proof. Of the sides of the ΔDEF, let EF be that which is *not less* than either of the other two.

Apply the ΔABC to the ΔDEF, so that BC, which = EF, may coincide with EF, and so that A may fall on that side of EF which is remote from D.

Let the ΔGEF be the new position of the ΔABC.

Join DG.

Then since, EG = ED, *Hyp.*
∴ the ∠EDG = the ∠EGD ; *Th. 5.*
and since FG = FD, *Hyp.*
∴ the ∠FDG = the ∠FGD. *Th. 5.*

Hence, the whole ∠EDF = the whole ∠EGF ; *Ax. 2.*
i.e. the ∠EDF = the ∠BAC.

Now in the Δ^s ABC, DEF, we have
AB = DE,
AC = DF,
and the included ∠BAC = the included ∠EDF ;
∴ the ΔABC = the ΔDEF in all respects. Q. E. D.

NOTE. In Theorem 7, it is proved that in the two Δs ABC and DEF, if it be *given* that AB=DE, AC=DF, and BC=EF, then it *necessarily* follows that $\angle$A=$\angle$D, $\angle$B=$\angle$E, $\angle$C=$\angle$F, and the area of the ΔABC=the area of the ΔDEF.

N. B. It must be remembered that *those angles are equal which are opposite to equal sides.*

EXERCISE 6

1. Prove that the straight line drawn from the vertex of an isosceles triangle to the mid-point of the base bisects the vertical angle and is also perpendicular to the base.

2. If P be a point equidistant from the ends of a straight line AB, and O, the middle point of AB, prove that PO is at right angles to AB.

3. ABC and DBC are two isos. Δs on the same base BC and on the same side of it. Prove that if AD be joined, and produced to meet the base, it will bisect the vertical angles of the Δs and be also perpendicular to the base.

4. ABC is an isos. Δ on the base BC ; if the bisectors of the base angles meet at D, and if AD be joined, prove that AD bisects the $\angle$BAC.

5. In the quadrilateral ABCD, AB=CD and AD=BC. Prove that $\angle$ABC=$\angle$ADC and $\angle$BAD=$\angle$BCD.

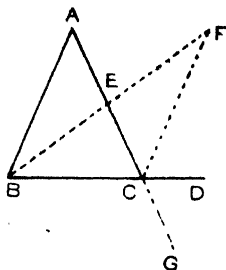
6. The sides of a quadrilateral ABCD are equal. Prove that the diagonal AC bisects the angles at A and C and the diagonal BD bisects the angles at B and D.

7. On the circumference of a circle, points A, B, C are taken such that AB and BC are equal. If O be the centre of the circle, prove that OB bisects the $\angle$AOC.

8. Two circles with centres A and B meet at the points C and D. Prove that AB bisects CD at right angles.

THEOREM 8.

If one side of a triangle be produced, then the exterior angle is greater than either of the interior opposite angles.



Let one side BC of the △ ABC be produced to D.

It is required to prove that the exterior ∠ ACD *is greater than either of the interior opposite* ∠ˢ BAC, ABC.

Let E be the middle point of AC.

Join BE; and produce it to F. making EF = BE.

Join CF.

Proof. In the △ˢ AEB, CEF, we have

AE = CE,

EB = EF,

and the ∠ AEB = the vertically opp. ∠ CEF.

∴ the two △ˢ are congruent.

Hence, the ∠ BAE = the ∠ ECF, being opp. to the equal sides BE, EF.

But the ∠ ECD is greater than the ∠ ECF;

∴ the ∠ ECD is greater than the ∠ BAE;

i.e., the ∠ ACD is > the ∠ BAC.

In the same way, producing AC to G and joining A to the middle point of BC, it may be proved that the ∠ BCG is greater than the ∠ ABC.

But the ∠ BCG = the vertically opp. ∠ ACD;

∴ the ∠ ACD is > the ∠ ABC. Q. E. D.

COR. 1. *Any two angles of a triangle are together less than two right angles.*

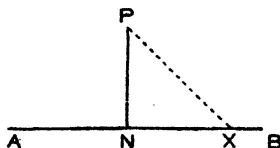
For, in the above diagram, the ∠ABC is less than the ∠ACD; ∴ the ∠ABC + the ∠ACB is less than the ∠ACD + the ∠ACB, and is ∴ < two rt. ∠s.

COR. 2. *Only one perpendicular can be drawn to a given straight line from a given point outside it.*

For, if PN, PX be two perpendiculars from P upon AB, each of the ∠s PXB, PNX is a rt. ∠.

But the ext. ∠PXB of the △PNX

> ∠PNX



i.e. a rt. ∠ > a rt. ∠, which is impossible.

COR. 3. *In every triangle, there must be at least two acute angles.*

This easily follows from Cor. 1.

EXERCISE 7

1. The two ∠s of any △ are together less than two right angles. Prove this by joining any point of a side to the opposite vertex.

2. Prove that in a triangle one angle of which is either right or obtuse, *each* of the other two angles is *acute*.

3. If in a △ABC, the ∠BAC be greater than the ∠ABC, prove that the angle ABC is acute.

4. D is any point within a △ABC; if BD and CD are joined, prove that the angle BDC is greater than the angle BAC.

5. Prove that from a given point outside a given straight line there can be drawn to it two and *only two* straight lines that are equal to one another.

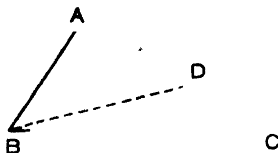
6. The ∠BAC of the △ABC is bisected by AD meeting BC at D. Prove that ∠ADC > ∠DAC and ∠ADB > ∠DAB.

7. Prove that the sum of any two exterior angles of a triangle is greater than two right angles.

8. Prove that the base angles of an isosceles triangle must be acute.

THEOREM 9.

If one side of a triangle be greater than another, then the angle opposite to the greater side is greater than the angle opposite to the less.



Let ABC be a $\triangle$, having the side AC > the side AB.

It is required to prove that the $\angle$ ABC *is* > *the* $\angle$ ACB.

From AC cut off AD = AB.

Join BD.

Proof. Because AB = AD,

$\therefore$ the $\angle$ ADB = the $\angle$ ABD.

But the ext. $\angle$ ADB of the $\triangle$ BDC is greater than the int. opp. $\angle$ DCB, *i.e.*, greater than the $\angle$ ACB.

$\therefore$ the $\angle$ ABD is > the $\angle$ ACB.

Much more, then, is the $\angle$ ABC greater than the $\angle$ ACB.

Q. E. D.

EXERCISE 8

1. If ABCD is a quadrilateral such that AB is the greatest and CD is the least of its sides, prove that $\angle$ BCD > $\angle$ BAD and $\angle$ ADC > $\angle$ ABC.

2. If the sides of a triangle are unequal, so are its angles.

3. The angles at the extremities of the greatest side of a triangle are acute.

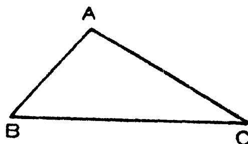
4. In a triangle, the angle opposite to the greatest side is the greatest.

5. AB, AD are the equal sides of an isosceles triangle ABD. The side AD is produced to any point C and BC is joined. Prove that $\angle$ ABC > $\angle$ ACB.

THEOREM 10.

[CONVERSE OF THEOREM 9]

If one angle of a triangle be greater than another, then the side opposite to the greater angle is greater than the side opposite to the less



Let ABC be a △, having the ∠ ABC greater than the ∠ ACB.

It is required to prove that the side AC > *the side* AB.

Proof. If AC is not greater than AB, then it must be *either* = AB, *or* less than AB,

(1) If AC = AB,

then the ∠ ABC = the ∠ ACB ;

which is impossible, by hypothesis.

(2) If AC be < AB,

then the ∠ ABC is less than the ∠ ACB,

which is also impossible, by hypothesis.

Thus, AC is neither equal to, nor less than AB ;

∴ AC > AB. Q. E. D.

EXERCISE 8A

1. ABC is a △, AD bisects the ∠ BAC and meets BC at D. Prove that AB > BD, and CA > CD. Hence show that AB + AC > BC.

2. Prove that in a right-angled triangle the hypotenuse is the greatest side.

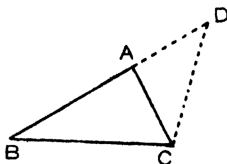
3. Prove that in a triangle, the side opposite to the greatest angle is the greatest.

4. If the angles of a triangle are unequal, so are the sides.

5. AB, AD are two equal sides of a △ABD ; if AD is produced to any point C and BC is joined, prove that DC is < BC. Hence deduce a construction for proving that the difference of any two sides of a triangle is less than the third side.

THEOREM 11.

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle.

It is required to prove that any two of its sides, AB *and* AC, *are together greater than the third side* BC.

Produce BA to D, making AD = AC.
Join DC.

Proof. Because AD = AC.
∴ the ∠ACD = the ∠ADC.

But the ∠BCD is > the ∠ADC;
∴ the ∠BCD is also > the ∠ADC,
that is, the ∠BCD is > the ∠BDC.

Hence, from the △BDC, we have
BD greater than BC.

But BA and AC are together = BD;
∴ BA and AC are together > BC.

Q. E. D.

COR. *The difference of any two sides of a triangle is less than the third side.*

For, AB − AC is < BC, if AB < AC + BC, and this latter is true.

[For a direct proof of the Corollary, see Ex. 1 of Exercise (9).]

NOTE. A straight line may be regarded as *the shortest distance between its extreme points.* From this point of view, the above theorem is self-evident.

EXERCISE 9

1. *The difference of any two sides of a triangle is less than the third side.*

[It can be proved directly as follows.

Make a construction as in the figure of Th. 9; and prove DC < BC, *i.e.*, AC − AB < BC.]

2. D is any point within a △ABC; if BD and CD are joined, prove that AB + AC > BD + DC.

3. If O is any point within a △ABC, prove that the sum of the lines OA, OB, OC is less than the sum, but greater than half the sum, of the sides of the triangle.

4. P is any point within the equilateral triangle ABC. Prove that the sum of any two of the straight lines PA, PB, PC is greater than the third.

5. If A is the greatest angle of the △ABC, show that AB + AC is > BC but < 2BC, and deduce that it is not possible to construct a triangle with sides equal to 2BC, CA and AB.

6. If D is the middle point of the side BC of the △ABC, prove that AB + AC > 2AD.

[*Hint.* Produce AD to E such that AD = DE and join EC. Then EC = AB. But EC + AC > AE, *i.e.* AB + AC > 2AD.]

Deduce that *in any triangle the sum of the medians is less than the perimeter.*

7. Show that any three sides of a quadrilateral are together greater than the fourth side.

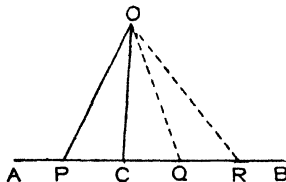
8. Prove that the sum of the diagonals of any quadrilateral is less than the sum, but greater than half the sum, of the sides of the quadrilateral.

9. If any point be taken within a quadrilateral, prove that the sum of its distances from the vertices is greater than half the sum of the sides of the quadrilateral.

10. Find the point within a quadrilateral such that the sum of its distances from the vertices is the least. [The pt. of intersection of the diagonals is the reqd. pt.]

THEOREM 12.

Of all the straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.



Let OC be the perpendicular from the given point O to the given str. line AB.

Let OP be *any other* str. line from O meeting the given str. line at P.

It is required to prove that OC < OP.

Proof. Because the side PC of the △OCP is produced to B,
∴ the int. opp. ∠OPC < the ext. ∠OCB.

But the ∠OCB = the ∠OCP, each being a right angle.
∴ the ∠OPC < the ∠OCP.
Hence OC < OP.

Q. E. D.

COR. 1. *Two straight lines* OP, OQ *which meet* AB *at equal distances from the foot of the perpendicular are equal.*

For, let OP, OQ meet AB at P and Q such that CQ = CP.

Now, in the △ˢ OCP and OCQ,
CP = CQ,
CO common,
and the ∠OCP = the ∠OCQ;
∴ the two triangles are congruent.
Hence OQ = OP.

COR. 2. *Of the two straight lines* OQ, OR, *if* OR *cuts* AB *at a greater distance from the foot of the perpendicular, then* OR *is greater than* OQ.

For, let CR > CQ.

Then, in the △OCQ, the ext. ∠OQR is > the int. opp. ∠OCQ;

∴ the ∠OQR is *obtuse*.

Also in the △OCR, the ext. ∠OQC is > the int. opp. ∠ORC;

∴ the ∠ORC is *acute*.

Hence the ∠OQR is > the ORQ;

∴ OR < OQ.

Q. E. D.

The slanting lines OP, OQ, OR are called obliques from O to AB.

DEFINITION

55. The length of the perpendicular from a given point upon a given straight line is called the **distance** of the point from that line.

EXERCISE 10

1. Of all straight lines drawn from a given point O to a given straight line AB, the straight line OC is the shortest. Prove that OC is perpendicular to AB.

2. AC is the hypotenuse of the right-angled triangle ABC. If points D and E be taken on BC and BC produced respectively, prove that AC > AD but < AE.

3. Prove that the median is the shortest of all lines that can be drawn from the vertex to the base of an isosceles triangle.

4. ABC is an equilateral triangle. Show that the shortest line that can be drawn from A to BC bisects the ∠BAC.

5. In a triangle ABC, the ∠ACB is the greatest. If N be the foot of the perpendicular from A on BC, prove that BN > CN.

6. BC is the greatest side of the △ABC. If AN is drawn perpendicular on BC prove that AB > BN and AC > CN.

Hence deduce Theorem 11.

CHAPTER IV

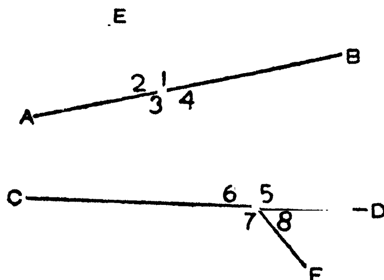
PARALLEL LINES

56. Straight lines which are in the same plane and which do not meet, however far they may be produced either way, are called **parallel straight lines.**

NOTE. It should be noted that parallel lines *must lie in the same plane* and *must not meet even when produced both ways.* Two straight lines which do not meet are not necessarily parallel. For example, a horizontal line and a vertical line at some distance from it would never meet ; but since they cannot be made to lie *in the same plane*, they are not parallel.

Two straight lines which cannot be made to lie in the same plane are called **skew lines.**

57. Any straight line which cuts two or more given lines is called a **transversal.**



Thus, EF is a transversal.

When a transversal meets two straight lines, altogether *eight* angles are formed. Of these

(i) the angles 1, 2, 7, 8 are called **exterior angles ;**

(ii) the angles 3, 4, 5, 6 are called **interior angles ;**

(iii) angles 3 and 5 are called **alternate angles.** Also 4 and 6 are *alternate angles* ;

(iv) angles 4 and 5 are called **interior angles on the same side of the transversal** EF. So also angles 3 and 6 are *interior angles on the same side of* EF ;

(v) angles 1 and 5 are said to be **corresponding angles.** Of these 1 is called the **exterior angle** and 5, **the interior opposite angle on the same side** of EF.

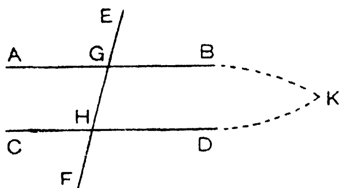
So also 2 and 6, 7 and 3, 8 and 4 are pairs of *corresponding angles.*

THEOREM 13.

If a straight line, cutting two other straight lines makes,

(i) *the alternate angles equal ;*
or (ii) *an exterior angle equal to the interior opposite angle on the same side of it ;*
or (iii) *the interior angles on the same side equal to two right angles ;*

then those two straight lines are parallel.



Let the str. line EF cut the two str. lines AB, CD at G and H.

(i) Let the alternate $\angle^s$ AGH, GHD be equal to one another.

It is required to prove that AB *and* CD *are parallel.*

Proof. If AB and CD are *not* parallel, they will meet when produced either towards B and D, or towards A and C.

Suppose, when produced towards B and D, they meet at the point K.

Then KGH is a Δ, of which the side KG has been produced to A ;

$\therefore$ the ext. $\angle$AGH is $>$ the int. opp. $\angle$GHK.

But this is contrary to the hypothesis ;

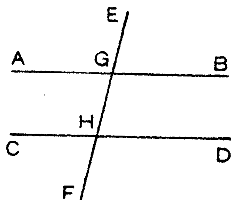
$\therefore$ AB and CD cannot meet when produced towards B and D.

Similarly they can neither meet towards A and C.

Hence AB and CD are parallel.

(ii) Let the ext. ∠EGB be = the int. opp. ∠GHD.

It is required to prove that AB *and* CD *are parallel.*



Proof. Because the ∠EGB = the ∠GHD,
and the ∠EGB = the vertically opp. ∠AGH;
∴ the ∠AGH = the ∠GHD;
and these are alternate angles.
∴ AB and CD are parallel.

(iii) Let the two int. ∠ˢ BGH, GHD, on the same side of EF, be together = two rt. ∠ˢ.

It is required to prove that AB *and* CD *are parallel.*

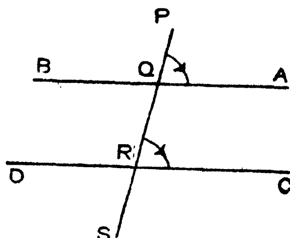
Proof. Because the ∠BGH + the ∠GHD
= two rt. ∠ˢ,
and also the ∠BGH + the ∠AGH
= two rt. ∠ˢ;
∴ the ∠BGH + the ∠GHD
= the ∠BGH + the ∠AGH.

Hence, the ∠GHD = the ∠AGH;
and these are *alternate angles.*
∴ AB and CD are parallel. Q. E. D.

COR. *If two straight lines are perpendicular to the same straight line, they are parallel to one another.*

NOTE. There are three different hypothesis, and in each case the conclusion is one and the same. Hence the proposition might very well be split up into three separate propositions.

58. Parallel lines have same direction.

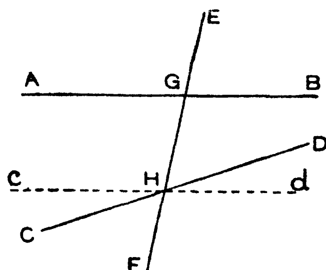


If any fixed and unlimited str. line PQRS cuts any two str. lines AB, CD at Q and R, AB is ∥ to CD when the ∠s AQP and CRP are *equal.* Hence if AB and CD be both supposed to be initially on the fixed line PS, and then to turn in the *watch-hand* way, AB about the point Q and CD about the point R, they will become parallel when they have undergone the *same amount of turning.* Hence, two str. lines are parallel when they have the *same direction.*

NOTE. In ordinary language we would say that the lines AB and BA have opposite directions ; but in Geometry they are said to have the same direction but opposite **senses** in that direction.

Since, CD is parallel to AB as well as to BA, it follows that two parallel lines must have the same direction ; but some or opposite *sense.*

59. Playfair's Axiom.



Let a str. line EF cut a str. line AB at G. Let another str. line OD rotate about the pt. H on EF, in the direction of the arrow-head, starting from a position of coincidence with EF. Then it is evident that the ∠EHD gradually increases, whilst the ∠DHF gradually decreases. Hence there is one position of CD *viz., cd,* and *one only* in which the ∠EHD = the ∠EGB, and in this position CD is parallel to AB.

Consequently, through a point H one and *only one* straight line *viz., cd* can be parallel to AB *i. e.,*

Through a given point one and only one straight line can be drawn parallel to a given straight line.

Hence, follows the **Playfair's Axiom** *viz.*,

Two intersecting straight lines cannot both be parallel to the same straight line.

For, if two intersecting lines, CD and *cd* say be parallel to the same straight line AB, then through the point H we shall have two straight lines parallel to AB which is impossible.

60. Hypothetical Construction.

From the above, it follows that the following geometrical construction also may be assumed for establishing a geometrical truth, *viz.*

Through any given point a straight line may be drawn parallel to a given straight line.

See Art. 35.

THEOREM 14.

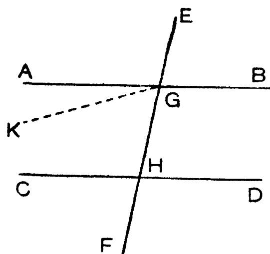
[CONVERSE OF THEOREM 13]

If a straight line cuts two parallel str. lines, it makes,

(i) *the alternate angles equal to one another ;*

(ii) *the exterior angle equal to the interior opposite angle on the same side of it ;*

and (iii) *the two interior angles on the same side together equal to two right angles.*



Let the str. line EF cut the ∥ str. lines AB, CD at the pts. G and H.

(i) *It is required to prove that the* ∠AGH = *the alternate* ∠GHD.

Proof. If the ∠AGH be not equal to the ∠GHD, let the ∠KGH be equal to it.

Now since the ∠ˢ KGH and GHD are alt. ∠ˢ, and equal to one another.

∴ KG is parallel to CD.

But, by hypothesis, AB is parallel to CD ;

∴ the two intersecting str. lines AG, KG, are *both* ∥ to CD, which is impossible. *Playfair's Axiom.*

Thus, the ∠AGH cannot be unequal to the ∠GHD ;

∴ ∠AGH = the ∠GHD.

(*ii*) *It is required to prove that*
the ext. ∠EGB = *the int. opp.* ∠GHD.

Proof. Because the ∠AGH = the ∠GHD,
already proved.

and also the ∠AGH = the vertically opp. ∠EGB;
∴ the ∠EGB = the ∠GHD.

(*iii*) *It is required to prove that the two interior* ∠s BGH, GHD *are together = two rt.* ∠s.

Proof. Because the ∠EGB = the ∠GHD,
proved above.

∴ the ∠EGB + the ∠BGH = the ∠GHD + the ∠BGH.

But the ∠s EGB, BGH are together = two rt. ∠s,
∴ the ∠s BGH, GHD are together = two rt. ∠s.
Q. E. D.

COR. *Two angles having parallel arms are either equal or supplementary.*

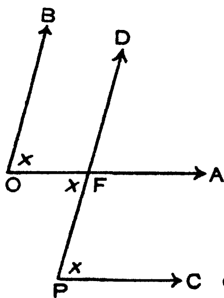


Fig. 1.

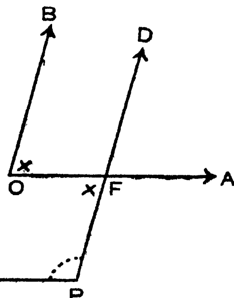


Fig. 2.

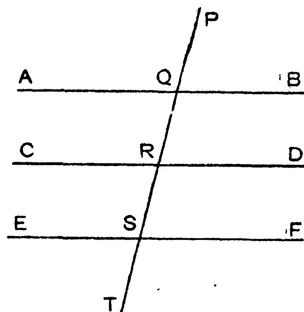
The arms OA, OB of the ∠AOB are parallel respectively to the arms PC, PD of the angle ∠CPD.

It is easy to prove that in fig. 1, the ∠AOB = the ∠CPD; whilst in fig. 2, the ∠AOB is supplementary to the ∠CPD.

NOTE. Notice that in fig. 1, arms OA and PC have the same sense, but in fig. 2, their senses are opposite.

THEOREM 15.

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to EF.

It is required to prove that AB *is parallel to* CD.

First Method : Draw any straight line PT cutting AB, CD and EF in the points Q, R and S respectively.

Proof. Since AB is ∥ to EF, and PT cuts them,

∴ the ∠AQS = the alt. ∠QSF.

Again, since CD is ∥ to EF, and PT cuts them,

∴ the ext. ∠QRD = the int. opp. ∠QSF.

Hence the ∠AQR = the ∠QRD.

But these are alternate ∠s;

∴ AB is ∥ to CD. Q. E. D.

Second Method : If AB is not ∥ to CD, they will meet when produced; and then we have two intersecting str. lines both parallel to a third straight line, which is impossible.

Playfair's Axiom.

Hence AB and CD cannot meet, and ∴ they are parallel.

Q. E. D.

NOTE. If EF lies *between* AB and CD, the truth of the theorem becomes obvious. For, if AB and CD were to meet, one of them would cut EF, which is impossible. Hence AB and CD cannot meet, and ∴ they are parallel.

EXERCISE 11

(*On Theorems 13-15*)

1. If a straight line intersecting two other straight lines make two exterior angles on the same side of the line together equal to two right angles, prove that the two straight lines are parallel.

2. Two straight lines which are perpendicular to a third straight line are parallel to each other.

3. In the triangle ABC, AB = AC. A straight line DE meets AB and AC at D and E such that $\angle$ADE = $\angle$ACB. Prove that DE is parallel to BC.

4. If a quadrilateral ABCD be such that the diagonals AC, BD bisect each other, prove that the opposite sides of the quadrilaterals are parallel.

5. If in the diagram of Th. 15, $\angle$AQP = 120°, find the remaining eleven angles.

6. If a str. line cuts two parallel str. lines, prove that the bisectors of any pair of alt. angles are parallel.

7. ABC is an isosceles triangle. DE is drawn parallel to the base BC cutting the sides AB, AC at D and E. Prove that the ΔADE is also isosceles.

8. DE is drawn parallel to the side BC of any ΔABC and meets the sides AB, AC at D and E. Prove that the Δs ABC, ADE are equiangular.

9. Through each angular point of a triangle a str. line is drawn parallel to the opposite side ; prove that the triangle formed by these three str. lines is equiangular to the given triangle.

10. If through the vertex of an isosceles triangle a straight line be drawn parallel to the base, prove that it bisects the exterior vertical angle.

11. If the side BC of a triangle ABC be produced to D, show that the bisectors of the angles BAC, ACD cannot be parallel.

12. If two straight lines be such that one of them is perpendicular to a given straight line while the other is not, prove that they must intersect each other.

13. If a straight line is perpendicular to one of two parallel str. lines it is also perpendicular to the other.

14. If any two straight lines be drawn perpendicular to two given intersecting straight lines, prove that they must intersect.

15. The opposite sides of a quadrilateral are parallel. Prove that

    (i) if one of its angles be a right angle, the other three angles also are right angles ;

    (ii) its opposite angles are equal ;

    (iii) the sum of its angles = 360°.

CHAPTER V

PROBLEMS

The preceding theorems can be applied to solve some easy problems, to which we shall now pass on.

61. A problem is a proposition in which certain geometrical construction is proposed and performed. The construction is to be *actually traced out* in each case by the student and *the only geometrical instruments to be used* for the purpose *are* (i) *a flat ruler* and (ii) *a pair of compasses.*

Figures drawn must be strictly accurate and all lines necessary for the construction must be shewn clearly.

A formal proof is appended to every problem to indicate that the proposed construction has been correctly effected. But the accuracy of the drawing should also be tested by *actual measurement.*

Any additional lines, which may be necessary for supplying the proof, *should be dotted* to distinguish them from the lines required for the construction.

For numerical applications of the problems and also for verifying the constructions mentioned above, the student should, however, provide himself with the following instruments :

(1) *Dividers.*

(2) *Pencil Compasses.*

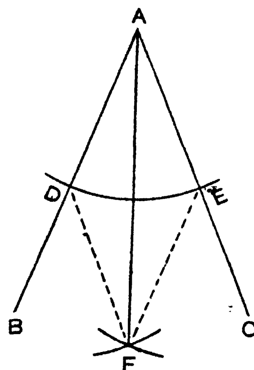
(3) *Two Set-Squares* ; one with angles of 45° and the other with angles of 60° and 30°.

(4) *A Graduated Flat Ruler* ; shewing inches and tenths of an inch on one side, and centimetres and tenths of a centimetre on the other.

(5) *A semi-circular protractor.*

PROBLEM 1.

To bisect a given angle.



Let BAC be the given angle.

It is required to divide it into two equal parts.

Construction. With centre A, and any radius, describe an arc of a circle cutting AB, AC at D and E respectively.

With centre D, and radius DE, describe an arc of a circle, on the side of DE remote from A; also with centre E, and radius ED, describe another arc cutting the former at F.

Join AF.

Then AF is the bisector of the ∠BAC.

Proof. Join DF, EF.

In the Δ' DAF, EAF, we have

AD = AE, (radii of the same circle)

DF = EF, (each being = DE).

and AF common ;

∴ the two Δ' are equal in all respects.

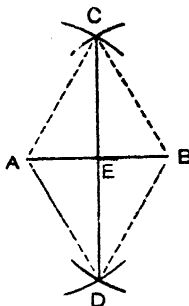
Hence the ∠DAF = the ∠EAF ;

that is, the ∠BAC is bisected by AF. Q. E. F.*

*The letter Q. E. F. at the end of a problem stand for the Latin words *Quod Erat Faciendum*, that is, *which was to be done.*

PROBLEM 2.

To bisect a given finite straight line.



Let AB be the given finite str. line.

It is required to divide it into two equal parts.

Construction. With centre A, and radius AB, describe two arcs, one on each side of AB.

With centre B, and radius BA, describe two other arcs cutting the former at C and D.

Join CD, cutting AB at E.

Then E is the middle point of AB.

Proof. Join AC, BC, AD, BD.

In the Δ^s ACD, BCD, we have

AC = BC, (each being = AB)

CD common,

and AD = BD, (each being = AB)

∴ the two Δ^s are equal in all respects.

Hence the ∠ACD = the ∠BCD.

Again, in the Δ^s ACE, BCE, we have

AC = BC,

CE common,

and the ∠ACE = the ∠BCE ; *proved above.*

∴ the two Δ^s are equal in all respects.

Hence AE = EB ;

i.e., AB is bisected at E. Q. E. F.

NOTE. This proposition furnishes a method of drawing a straight line bisecting a given straight line at right angles. Thus, in the above figure the straight line CD is the perpendicular bisector of AB.

EXERCISE 12

1. Draw angles on a sheet of paper and bisect them. Test the accuracy of the constructions with a protractor.

2. Divide an angle into four equal parts. Hence, divide an angle into two parts one of which is three times the other. Check your result with a protractor.

3. Divide an angle into two parts one of which is seven times the other. Verify the result by measurement.

4. Draw an acute angle and construct its internal and external bisectors. Find the angle between the two bisectors with a protractor.

5. Draw a triangle on a sheet of paper and bisect its angles. Do these bisectors meet ?

6. Draw straight lines and bisect them. Check the result in each case by measurement.

7. Divide a straight line into 4 equal parts. Hence divide a straight line into two parts one of which is 3 times the other. Verify the result with the dividers.

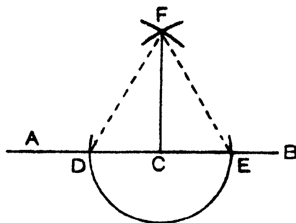
8. Divide a straight line into 2 parts such that one is 7 times the other.

9. Draw a triangle. Trace the perpendicular bisector of its sides. Do they meet ?

10. Trace the medians of a triangle. Do you find them concurrent ?

PROBLEM 3.

To draw a straight line perpendicular to a given straight line from a given point in it.



Let AB be the given str. line. and C the given point in it.

It is required to draw through C *a str. line* ⊥ *to* AB.

Construction. With centre C, and any convenient radius, describe a circle cutting AB at D and E.

With centres D and E, and radius DE, describe two arcs cutting each other at F.

Join CF.

Then CF is perpendicular to AB.

Proof. Join FD, FE.

In the Δ^s FDC, FEC, we have

FD = FE, (each being = DE)

DC = EC, *Cons.*

and FC common ;

∴ the two Δ^s are equal in all respects.

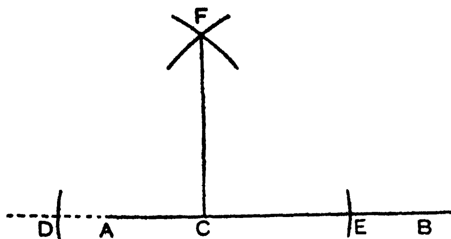
Hence the ∠FCD = the ∠FCE.

But these are adjacent angles ; hence each of them is a right angle.

∴ CF is ⊥ to AB.

Q. E. F.

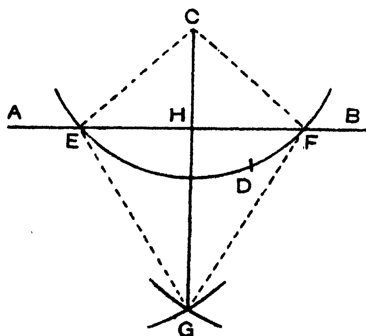
NOTE. If C is close to an extremity of AB, the line AB may be produced as in the figure below.



N. B. For other methods of construction, see Ex. 14 and 15 of Exercise (14).

PROBLEM 4.

To draw a straight line perpendicular to a given straight line from a given point outside it.



Let AB be the given str. line and C the given pt. outside it.

It is required to draw from C *a straight line* ⊥ *to* AB.

Construction. Take any point D on the side of AB remote from C.

With centre C, and radius CD, describe a circle cutting AB at E and F.

With centres E and F, and radius EF, describe two arcs, on the side of AB remote from C, cutting each other at G.

Join CG cutting AB at H.
Then CH is ⊥ to AB.

Proof. Join CE, CF, GE, GF.

In the Δ^s CEG, CFG, we have

CE = CF,

EG = FG, (each being = EF)

and CG common;

∴ the two Δ^s are equal in all respects.

Hence the ∠ECG = the ∠FCG.

Again in the Δ^s CEH, CFH, we have

CE = CF,

CH is common,

and the ∠ECH = the ∠FCH;

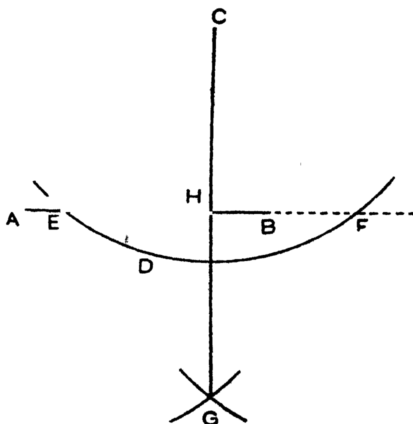
∴ the two Δ^s are equal in all respects.

Hence the ∠CHE = the ∠CHF; and these are adjacent angles.

∴ each of them is a rt. angle.

∴ CH is ⊥ to AB. Q. E. F.

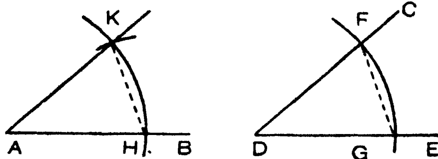
NOTE. If the point C is nearly opposite to one end of AB, the line AB may be produced as in the diagram below.



N. B. For another method, see Ex. 16 of Exercise (14).

PROBLEM 5.

At a given point in a given straight line to make an angle equal to a given angle.



Let A be the given pt. in the given str. line AB, and CDE the given angle.

It is required to make at the pt. A *in the str. line* AB *an angle equal to the* ∠CDE.

Construction. With centre D, and any radius, describe a circle cutting DC and DE at F and G respectively.

With centre A, and radius DG, describe an arc cutting AB at H; with centre H, and radius GF, describe an arc cutting the former at K.

Join AK.

Then KAH will be the reqd. angle.

Proof. Join FG, KH.

In the Δ^s KAH, FDG, we have

AH = DG, *Cons.*

AK = DF, *Cons.*

and HK = GF; *Cons.*

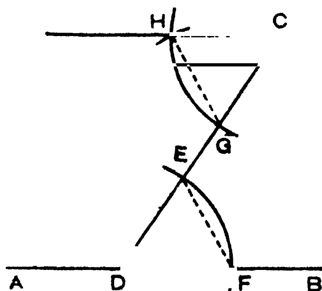
∴ the two Δ^s are equal in all respects.

Hence the ∠KAH = the ∠FDG.

Q. E. F.

PROBLEM 6.

Through a given point to draw a straight line parallel to a given straight line.



Let C be the given point, and AB the given straight line.

It is required to draw through C *a straight line parallel to* AB.

Construction. Take any point D in AB, and join CD.

With centre D, and any radius, describe a circle cutting CD at E and DB at F.

With centre C, and radius DE, describe an arc, on the side of CD remote from B, cutting CD at G.

With centre G, and radius FE, describe an arc cutting the former at H.

Join CH.

Then CH is parallel to AB.

Proof. Join EF, HG.

In the Δ^s CHG, DFE, we have

	CH = DF,	*Cons.*
	CG = DE,	*Cons.*
and	GH = FE ;	*Cons.*

∴ the two Δ^s are equal in all respects.

Hence, the ∠GCH = the ∠EDF and these are alternate angles ;

∴ CH is parallel to AB. Q. E. F.

EXERCISE 13

1. Take any point on a sheet of paper and at it construct an angle of (i) 90°, (ii) 45° and (iii) 135° without using a protractor. Check the results with a protractor.

2. Draw a right-angled isosceles triangle, one of the equal sides being 2 inches. Find the angles of the triangle by measurement.

3. Bisect an angle AOB by OC and draw OD at right angles to OC. Show by measurement that OD bisects the angle AOB externally.

4. Construct a triangle whose base is 3 *cms.* long and whose base angles are each 45°. Measure its vertical angle with a protractor.

5. Draw a circle and take two points A and B on it. Construct the perpendicular from the centre on AB and show by measurement that AB is bisected by this perpendicular.

6. Construct any triangle and draw perpendiculars from the vertices to the opposite sides. Do they meet in a point?

7. From any two points P and Q outside a given a straight line AB, draw lines equally inclined to AB.

8. The angle ABC is a rt. angle. Through any point D within the angle draw a str. line parallel to AB and BC. Show by measurement that opposite sides of the quadrilateral thus constructed are equal.

(*On Problems 5 and 6*)

9. Construct angles equal to given angles and check your result by measurement.

10. Construct the supplement of the angle AOB (i) at O and (ii) at some other point P.

11. Given an acute angle AOB. Construct its complement (i) at O and (ii) at another point P.

12. Find the two angles whose sum and difference are the two given angles ABC and PQR.

13. Draw a triangle whose base is 5 *cms.* and angles at the base equal to two given angles.

14. Through given points draw lines parallel to a given line.

15. From a given point outside a given straight line, draw a line making an angle with the given line equal to a given angle.

16. Construct a triangle of which the base angles shall be equal to two given angles and the perpendicular from the vertex to the base equal to a given straight line.

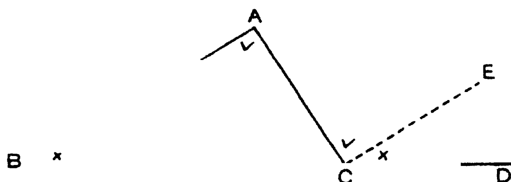
CHAPTER VI

THEOREMS

1. Angles in a rectilineal figure

THEOREM 16.

The sum of the angles of a triangle is equal to two right angles.



Let ABC be a triangle.

It is required to prove that $\angle$BCA + $\angle$CAB + $\angle$ABC = 2 *right angles.*

Produce BC to any point D and suppose CE is drawn parallel to BA.

Proof. Because CE is par' to BA
and AC meets them ;
$\therefore$ the $\angle$ACE = the alternate $\angle$CAB.

Again, because CE and BA are par' and BD meets them ;
$\therefore$ the ext. $\angle$ECD = the int. opp. $\angle$BAC
$\therefore$ the whole $\angle$ACD = $\angle$CAB + $\angle$ABC...............(*a*)

To each of these equals add the $\angle$BCA
$\therefore$ $\angle$BCA + $\angle$ACD = $\angle$BCA + $\angle$CAB + $\angle$ABC.

But since BCD is a str. line.
$\angle$BCA + $\angle$ACD = 2 rt. angles.
$\therefore$ $\angle$BCA + $\angle$CAB + $\angle$ABC = 2 rt. angles.

COR. 1. *If any side of a triangle is produced, the exterior angle is equal to the sum of the two interior apposite angles.*

This important property has been directly deduced in the course of the above proof.

Thus, from (*a*), the exterior $\angle ACD = \angle CAB + \angle ABC$.

COR. 2. *If two angles of one triangle are respectively equal to two angles of another, the third angle of the one is equal to the third angle of the other.*

COR. 3. *The sum of any two angles of a triangle is less than two right angles.*

COR. 4. *If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.*

COR. 5. *In any right-angled triangle the two acute angles are complementary.*

NOTE. If A, B, C be the angles of a triangle in degrees,

$$A + B + C = 180^\circ.$$

Ex. 1. Each angle of an equilateral triangle $= 60^\circ$.

For, the angles of an equilateral triangles are equal to one another.

$\therefore$ If A denote any angle, we have

$$A + A + A = 180^\circ$$
$$\text{or,} \quad 3A = 180^\circ$$
$$\text{i.e.,} \quad A = 60^\circ.$$

Ex. 2. Each of the acute angles of a right-angled isosceles triangle is 45°.

The triangle being isosceles, the two acute angles are equal; also, their sum $= 90^\circ$, *Th. 16, Cor. 5.*

$\therefore$ Each of them $= 45^\circ$.

Ex. 3. Two angles of a triangle are 55° and 102°. Find the third angle.

If $A =$ the third angle, we have $A + 55^\circ + 102^\circ = 180^\circ$.

or, $A + 157^\circ = 180^\circ$; $\therefore$ $A = 180^\circ - 157^\circ = 23^\circ$.

EXERCISE 14

1. In an isosceles triangle, the vertical angle is 72°. Find the base angles.

2. Find the magnitude of the vertical angle of an isosceles triangle each of the base angles of which is double of the vertical angle.

3. Two angles of a triangle are 50° and 70°. Find the third angle.

4. If one angle of a triangle is double the sum of the other two, find it.

5. Of the three angles A, B, C of a triangle, B is the double of A and C is three times B. Find each.

6. In the figure of Th. 16,

(i) if $\angle$CAB=45° and $\angle$ABC=50°, find $\angle$ACD ;

(ii) if $\angle$ACD=125° and $\angle$ABC=48°, find $\angle$CAB ;

(iii) if $\angle$ACD=130° and $\angle$CAB=81°, find $\angle$ABC.

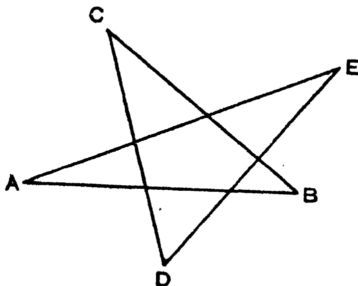
7. Prove that the sum of the angles of a quadrilateral is equal to four right angles.

8. If one angle of a triangle be greater than the sum of the other two, prove that the triangle is obtuse-angled.

9. If the exterior angles formed by producing one side of a triangle both ways be 106° and 125°, find all the angles of the triangle.

10. The three exterior angles formed by producing the sides of a triangle in order are together equal to four right angles.

11. Prove that the sum of the angles at A, B, C, D and E in the figure formed by the str. lines AB, BC, CD, DE and EA is two right angles. Apply *Cor. 1 of Th. 16.*

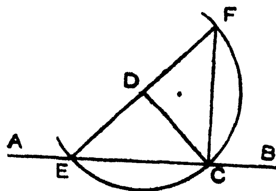


12. Prove that the straight line bisecting any of the angles exterior to the vertical angle of an isosceles triangle is par^l to the base.

13. If A is the vertex of an isosceles triangle ABC and BA is produced to D, so that AD=AB and if DC is joined, prove that BCD is a right angle.

14. Explain the following method of constructing a perpendicular to a given straight line AB from any point C in it.

Construction. Take any point D outside AB. With centre D, and radius DC, describe a circle cutting AB at E. Join ED, and produce it to meet the circle at F. Join CF.



Then CF is ⊥ to AB.

[**Proof.** Join DC.
Since DE=DC, ∴ the ∠DCE=the ∠DEC.
Also ∵ DF=DC, ∴ the ∠DCF=the ∠DFC.

∴ the whole ∠ECF=the ∠FEC+the ∠EFC
=the ext. ∠BCF.

Th. 16, Cor. 1.

But ∠s ECF, BCF are adjacent angles, ∴ each of them=a rt. ∠.
i.e., CF is ⊥ to AB.]

N. B. This method is applicable when C is near or at one end of AB.

15. Explain the following method of constructing a perpendicular to a given straight line AB from any pt. C on it.

Construction. With centre C and any radius, draw an arc of a circle cutting CB in D. With centre D and the *same* radius draw an arc cutting the first arc at E. With centre E and the same radius draw an arc cutting the first arc at F. With centre F and the same radius, describe an arc cutting the preceding arc at G. Join CF.

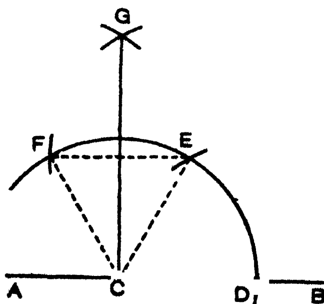
Then CG is ⊥ to AB.

[**Proof.** It is easy to see that ∠s DCE, ECF are angles of equilateral triangles DCE, ECF.

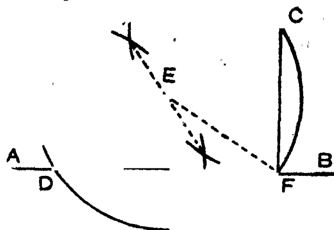
∴ ∠DCE=∠ECF=60°.
Also, CG bisects the ∠ECF,

Prob. 1.

∴ ∠ECG=30°.
∴ ∠DCG=∠DCE+∠ECG=90°.
i.e. CG ⊥ AB.]



16. Explain the following method of constructing a perpendicular to a given line AB from any external point C.



Construction. Take any point D in AB. Join DC and bisect it at E. With centre E, and radius EC, describe a circle cutting AB at D and F.

Join CF.
Then CF is ⊥ to AB.

[**Proof.** Join EF.

Since ED = EF, ∴ the ∠EFD = the ∠EDF.

Also since EC = EF, ∴ the ∠EFC = the ∠ECF,

∴ the whole ∠DFC = the ∠CDF + the ∠DCF

= the ext. ∠BCF. *Th. 16, Cor. 1.*

But the ∠s DFC, BFC are adjacent ∠s, and ∴ each of them is a right angle. Hence CF is ⊥ to AB.]

17. ABC, ABD, ACE are equilateral triangles. Prove that the points D, A, E are in the same straight line.

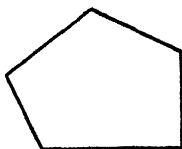
18. BC is the hypotenuse of a right-angled triangle ABC. If AD be drawn to meet BC, so that the ∠BAD is equal to the ∠ABD. Prove that D is the middle point of BC and that AD is half of BC.

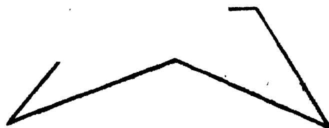
19. If two straight lines are perpendicular to two other straight lines, each to each, the acute angle between the first pair is equal to the acute angle between the second pair.

20. The angles ABC, ACB of △ABC are bisected by BO and CO. Prove that $\angle BOC = 90^\circ + \frac{1}{2} \times \angle CAB$.

21. The sides AB, AC of the △ABC are produced and the ext. ∠s at B and C are bisected by BO and CO. Prove that $\angle BOC = 90^\circ - \frac{1}{2} \angle CAB$.

62. A rectilineal figure is said to be **convex** when it has no re-entrant angle.



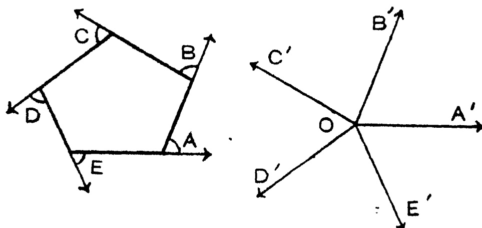


The first is a convex polygon but not tho second.

63. A **regular** polygon is one with all its sides equal and all its angles equal.

THEOREM 17.

If the sides of a convex rectilineal figure are produced in order, then the sum of the exterior angles so formed is equal to four right angles.



Let the sides of the rectilineal figure ABCDE be respectively produced in the directions shown by the arrow-heads.

It is required to prove that the exterior $\angle^s$ A, B, C, D, E *are together = four right* $\angle^s$.

Take any point O, and suppose the str. lines OA′, OB′, OC′, OD′, OE′ are drawn par^l respectively to the sides EA, AB, BC, CD, DE ; each being drawn in the *sense* in which the corresponding side is produced.

Proof. Since OA′ is par^l to EA produced, and OB′ is par^l to AB,

∴ the ext. $\angle$ A = the $\angle$ A′OB′.

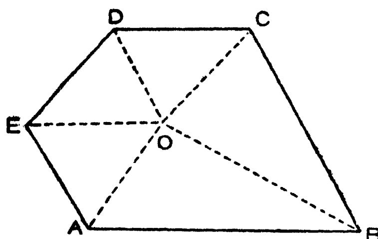
Similarly, { the ext. $\angle$ B = the $\angle$ B′OC′,
the ext. $\angle$ C = the $\angle$ C′OD′,
the ext. $\angle$ D = the $\angle$ D′OE′.
the ext. $\angle$ E = the $\angle$ E′OA′.

Hence, the sum of the ext. $\angle^s$ A, B, C, D, E
= the sum of the $\angle^s$ at O
= four right angles.

Q. E. D.

THEOREM 17A.

The sum of the interior angles of any rectilineal figure, together with four right angles is equal to twice as many right angles as the figure has sides.



Let ABCDE be a rectilineal figure of n sides.

It is required to prove that

the angles of the figure + 4 rt. $\angle^s = 2n$ rt. $\angle^s$.

Take any point O within the figure and join O to each of its vertices.

Proof. The figure has been divided into n triangles, and the sum of the three angles of each triangle = 2 rt. $\angle^s$.

$\therefore$ the sum of all the angles of the n triangles $= 2n$ rt. $\angle^s$.

But the sum of all the angles of the n triangles = the angles of the figure + the angles at O;

and the angles at O = 4 rt. $\angle^s$.

$\therefore$ the angles of the figure + 4 rt. $\angle^s = 2n$ rt. $\angle^s$.

Q. E. D.

64. Inferences from Theorems 17 and 17A.

1. Whatever be the number of sides of a convex rectilineal figure, the sum of its ext. $\angle^s$ is always = 4 rt. $\angle^s$, *i.e.* 360°.

2. If n be the number of sides of a regular rectilineal figure, each of its ext. $\angle = \frac{4}{n}$ rt. $\angle^s = \frac{360°}{n}$.

3. The sum of the interior angles of any rectilineal figure of n sides $=(2n-4)$ rt. $\angle^s$.

4. An interior angle of a regular rectilineal figure of n sides $=\frac{2n-4}{n}$ rt. $\angle^s = \frac{2n-4}{n} \cdot 90°$.

Ex. 1. Find the sum of the interior angles of a heptagon (a rect. figure of 7 sides).

The sum of the int. $\angle^s$ of a fig. of n sides $=(2n-4)$ rt. $\angle^s$.

$\therefore$ Putting, $n=7$, the sum of the int. $\angle^s$ of a heptagon $=(2\times 7-4)$ rt. $\angle^s = 10$ rt. $\angle^s = 900°$.

Ex. 2. If the sum of the angles of a rect. figure $=8$ rt. $\angle^s$, find the number of its sides.

Let $n=$ no. of sides reqd.

$\therefore$ The sum of the int. $\angle^s=(2n-4)$ rt. $\angle^s = 8$ rt. $\angle^s$

or, $2n-4=8$

or, $2n=8+4=12$,

$\therefore$ $n=6$.

Ex. 3. Find, in degrees, an interior angle of a regular pentagon.

A regular pentagon has got five equal int. $\angle^s$.

$\therefore$ All the five ext. $\angle^s = 4$ rt. $\angle^s = 360°$,

$\therefore$ each ext. $\angle = \frac{360°}{5} = 72°$.

Hence, each int. $\angle =$ supplement of the ext. $\angle = 180° - 72° = 108°$.

Otherwise thus :

An int. $\angle$ of a regular polygon of n sides $= \frac{2n-4}{n} \cdot 90°$.

$\therefore$ Putting $n=5$, we have an int. $\angle$ of a regular pentagon

$$= \frac{2\times 5-4}{5} \cdot 90° = \frac{6\times 90}{5} = 108°.$$

Ex. 4. An exterior angle of a regular polygon is 30°. Find the number of its sides.

If n = number of sides reqd. each ext. $\angle = \frac{360°}{n} = 30°$.

$\therefore$ The number of sides reqd. $= n = \frac{360}{30} = 12$.

Ex. 5. An interior $\angle$ of a regular polygon is 120°, find the number of its sides.

Evidently, an ext. $\angle$ of the fig. $= 180° - 120° = 60°$.

If n = no. of sides reqd., an ext. $\angle = \frac{360°}{n} = 60°$

$\therefore \quad n = \frac{360}{60} = 6$.

EXERCISE 15

1. Find the sum of the interior angles of a polygon of (i) 5 sides; (ii) 9 sides ; (iii) 8 sides ; (iv) 11 sides ; (v) 15 sides.

2. Given the sum of the interior angles of a rect. figure to be (i) 180° ; (ii) 360° ; (iii) 540° ; (iv) 720° ; (v) 20 rt. $\angle$s, find the number of the sides of the figure in each case.

3. Find an ext. $\angle$ of a regular polygon of (i) 9 sides ; (ii) 12 sides ; (iii) 15 sides ; (iv) 14 sides.

4. Given an ext. $\angle$ of a regular rect. figure to be (i) 20° ; (ii) 36° ; (iii) 40° ; (iv) 90° ; (v) 120°, find the number of its sides.

5. Given an int. $\angle$ of a regular rect. figure to be (i) 60° ; (ii) 90° ; (iii) 108° ; (iv) 144° ; (v) 150°, find the number of its sides.

6. Four angles of a pentagon are equal and are each double of the fifth angle. Find the angles.

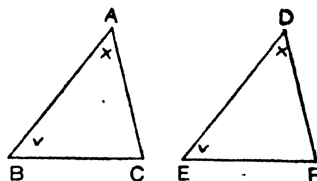
7. An ext. $\angle$ of a regular rect. figure is double of an int. $\angle$. Find the number of its sides.

8. The sum of the interior angles of a rect. figure is equal to the sum of its ext. $\angle$s. Find the number of its sides.

II. Congruence of Triangles—(*Contd.*)

THEOREM 18.

If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another, then these two triangles are congruent.



Let ABC, DEF be two triangles in which

$\angle A = \angle D$

$\angle B = \angle E$

and the side $BC =$ the corresponding side EF.

It is required to prove that the two Δ^s *are congruent.*

Proof. Because the sum of the angles of a $\Delta = 2$ rt. $\angle^s$,

$\therefore \quad \angle A + \angle B + \angle C = \angle D + \angle E + \angle F.$

But $\angle A = \angle D$ and $\angle B = \angle E$,

$\therefore \quad \angle C = \angle F.$

Apply the ΔABC to the ΔDEF so that B may be on E and BC on EF.

Then, because $BC = EF$,

$\therefore$ C must coincide with F.

Now, since the $\angle B =$ the $\angle E$,

$\therefore$ BA must fall on ED;

and since the $\angle C =$ the $\angle F$,

$\therefore$ CA must fall on FD.

Thus the point A falls both on ED and FD;

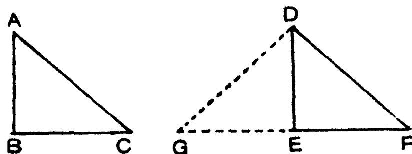
$\therefore$ A falls on D which is the point in which ED and FD intersect.

Hence the two Δ^s coincide, and are $\therefore$ congruent.

Q. E. D.

THEOREM 19.

Two right-angled triangles which have their hypotenuses equal, and one side of the one equal to one side of the other, are equal in all respects.



Let ABC and DEF be two right-angled Δ^s, in which the hypotenuse AC = the hypotenuse DF, and the side AB = the side DE.

It is required to prove that the two Δ^s are equal in all respects.

Proof. Apply the Δ ABC to the Δ DEF so that AB may coincide with DE to which it is equal, and C may fall on that side of DE which is remote from F.

Thus, let DEG be the new position of the Δ ABC.

Now, since each of the ∠^s DEF, DEG, is a rt. ∠,

∴ EG is in the same str. line with FE.

Hence, DGF is a Δ, and since DF = DG. *Hyp.*

∴ the ∠ DGF = the ∠ DFG.

Hence, in the Δ^s DEG, DEF, we have

the ∠ DEG = the ∠ DEF, being rt. ∠^s;

the ∠ DGE = the ∠ DFE, *proved above.*

and DE common;

∴ these two Δ^s are equal in all respects.

That is, the Δ^s ABC, DEF are equal in all respects.

Q. E. D.

EXERCISE 16

(On Theorem 18)

1. If the bisector of the vertical angle of a triangle is also perpendicular to the base, prove that the triangle is isosceles.

2. Prove that any point on the bisector of an angle is equidistant from the arms of the angle.

3. If a straight line AB is bisected at O, prove that the perpendiculars from A and B upon any other straight line through O are equal.

4. Prove that perpendiculars drawn from the extremities of the base of an isosceles triangle upon the opposite sides are equal.

5. ABCD is a quadrilateral such that the diagonal AC bisects each of the angles BAD, BCD. Prove that AC bisects BD at right angles.

(On Theorem 19)

6. Prove that the perpendicular from the vertex to the base of an isosceles triangle bisects the vertical angle and the base.

7. The distance of any point P from two lines AB, AC are equal. Prove that the str. line AP bisects the $\angle$BAC.

8. If the perpendiculars from the vertices B and C of the ΔABC upon opposite sides are equal, prove that the triangle is isosceles.

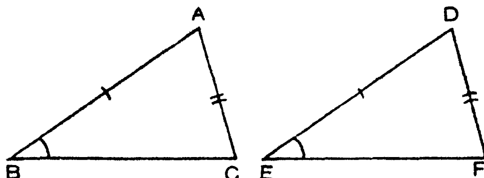
9. If A and B be two points taken on a circle, prove that the perpendicular from the centre on AB bisects AB.

10. If A, B, C, D be four points on a circle such that the perpendiculars from the centre on the str. lines AB and CD are equal, prove that AB=CD.

11. The bisectors of the angles at B and C of the ΔABC meet at O. Prove that AO bisects the $\angle$BAC.

THEOREM 20.*

If two triangles have two sides of the one respectively equal to two sides of the other, and have also the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary.



Let the Δ^s ABC, DEF be such that AB = DE, AC = DF, and the $\angle$ABC = the $\angle$DEF.

It is required to prove that the $\angle^s$ ACB, DEF *are either equal or supplementary.*

Proof. The angle BAC must be *either* equal to the $\angle$EDF *or* not.

Case I. When the $\angle$BAC = the $\angle$EDF.

In the Δ^s ABC, DEF, we have,

AB = DE,

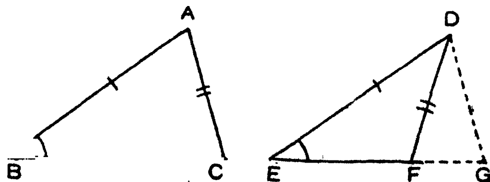
AC = DF,

and the $\angle$BAC = the $\angle$EDF;

$\therefore$ the two Δ^s are congruent.

Hence the $\angle$ACB = the $\angle$DFE,

Case II. When the $\angle$BAC is *not* equal to the $\angle$EDF.



On the same side of DE as F, suppose the $\angle$EDG is made equal to the $\angle$BAC. and let EF, produced if necessary, meet DG in G.

Then in the Δ^s ABC, DEG, we have
the $\angle$ABC = the $\angle$DEG,
the $\angle$BAC = the $\angle$EDG, *Cons.*
and AB = DE ;
$\therefore$ the two Δ^s are congruent.

Hence, DG = AC = DF ;
and also the $\angle$DGE = the $\angle$ACB,
$\therefore$ the $\angle$DFG = the $\angle$DGE [$\because$ DG = DF
= the $\angle$ACB.

But the $\angle$DFG is supplementary to the $\angle$DFE ;
$\therefore$ the $\angle$ACB is supplementary to the $\angle$DFE.

Q. E. D.

NOTE 1. It is possible for the angle ACB and DFE to be supplementary *only when one of them is acute and the other obtuse.*

NOTE 2. Clearly, the two triangles ABC, DEF are equal in all respects if $\angle$s ACB and DFE are equal but they are not so when $\angle$s ACB and DFE are supplementary. In the construction of triangles this is called the **Ambiguous Case**. The ambiguity disappears if $\angle$ABC or $\angle$DEF is a rt. $\angle$(See Th. 19) or an obtuse $\angle$, in which cases the triangles are clearly congruent.

EXERCISE 16A

1. If the perpendiculars from the middle point of the base of a triangle to the other two sides be equal, the triangle is isosceles.

2. ABC is a triangle and D is the middle point of the base BC ; if DE and DF are perpendiculars drawn from D to the sides AB and AC respectively and if BE = CF, prove that the triangle is isosceles.

3. The straight line drawn from the centre of a circle at right angles to a chord bisects that chord.

4. Any two equal chords of a circle are equidistant from its centre.

5. In a quadrilateral, ABCD, if the angles ABC, BCD be two equal obtuse angles and the diagonals AC, BD equal, prove that AB = CD.

6. Perpendiculars drawn from a point to the arms of an angle are equal. Show that the point is on the bisector of the angle.

NOTE ON THE CONGRUENCE OF TRIANGLES

65. Every triangle has six parts *viz., three* sides and *three* angles.

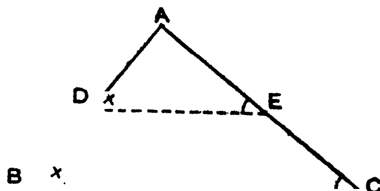
It has been shewn that triangles are equal in all respects if the following three parts of the one are equal to the corresponding three parts of the other.

1. *Two sides and the included angle.* (*Theorem* 4).
2. *Three sides.* (*Theorem* 7).
3. *Two angles and a corresponding side.* (*Theorem* 18).
4. *Two sides and an angle not included and not acute.* (*Theorems* 19 and 20).

[We have noticed in Th. 20 that if the angle is acute, the triangles *may or may not be congruent*—which is the ambiguous case.]

Two triangles are *not necessarily congruent* when three angles of the one are equal to the three angles of the other.

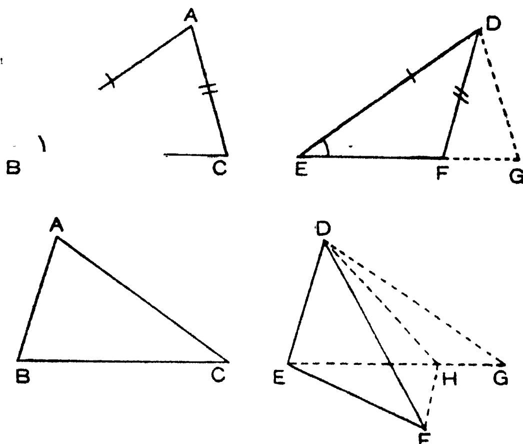
As for instance, if DE is par' to BC, the Δ^s ADE, ABC are equiangular but *not congruent.*



For the congruence of two triangles, we must have three parts of the one equal to the three parts of the other, and these must be chosen in one of the four ways mentioned above.

THEOREM 21.*

If two triangles be such that two sides of the one are respectively equal to two sides of the other, but the angle contained by the two sides of the one is greater than the angle contained by the two sides of the other; then the base of that which has the greater angle is greater than the base of the other.



Let ABC, DEF be two △s such that

AB = DE,

AC = DF,

but the ∠BAC is greater than the ∠EDF.

It is required to prove that the base BC *is greater than the base* EF.

Proof. Apply the △ABC to the △DEF so that A may fall on D and AB on DE.

Then since AB = DE, ∴ B coincides with E.

Let DG, EG be the new positions of AC, BC.

Since, ∠BAC > ∠EDF, DF must be within the ∠EDG.

Case I. If EG passes through F as in fig. (1), then EG is > EF ; *i.e.*, BC is > EF.

Case II. If EG does not pass through F, as in fig. (2), suppose the ∠ FDG is bisected by DH which meets EG at E.

Join FH.

Then in the Δ^s FDH, GDH, we have

FD = GD,

DH common,

and the ∠ FDH = the ∠ GDH ;

∴ the two triangles are congruent.

Hence, FH = GH,

and ∴ EG = EH + GH = EH + FH.

But the two sides EH, HF of the Δ EHF are together > EF ;

∴ EG > EF,

that is, BC > EF.

Q. E. D.

COR. *If two triangles* ABC, DEF *be such that* AB *is equal to* DE, AC *is equal to* DF, *but the base* BC *is greater than the base* EF *then the angle* BAC *is greater than the angle* EDF.

For, the ∠ BAC cannot be less than the ∠ EDF ; for then BC would be less than EF, which is impossible.

Nor can the ∠ BAC be equal to the ∠ EDF ; for then BC would be equal to EF, which also is impossible.

∴ the ∠ BAC *must be* > the ∠ EDF.

CHAPTER VII

PROBLEMS

CONSTRUCTION OF TRIANGLES

66. We have noticed that two triangles are equal in all respects (*i.e.*, exactly same in *shape* and *size*) if three parts of one triangle are equal to the corresponding three parts of the other, provided these parts are chosen as described in § 65.

Hence it follows that a triangle of a definite size and shape can be constructed with any one of the following combinations of three parts or with any three data equivalent to any one of these combinations :

(1) *The three sides* ;

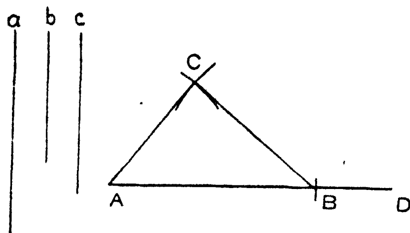
(2) *Two sides and the contained angle* ;

(3) *Two angles and a side* ;

(4) *Two sides and an angle opposite one of these sides, provided that the angle is not acute.* (If the angle be acute, two triangles can be constructed with such given parts as shown in Prob. 10 ; hence, the shape and size of the triangle are not definite. This is called the **Ambiguous Case**).

PROBLEM 7.

To construct a triangle having its sides equal to three given straight lines, any two of which are together greater than the third.



Let a, b, c be three given str. lines, of which a is the greatest, and any two of which are together greater than the third.

It is required to construct a triangle having its sides equal to the straight lines a, b, c.

Construction. Take any str. line AD, and from it cut off AB equal to a.

With centre A, and radius b, describe a circle.

With centre B, and radius c describe another circle cutting the former at C.

Join AC, BC.

Then ABC is the reqd. △ ; because, by construction,

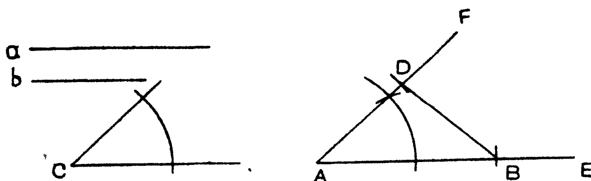
$$AB = a,$$
$$AC = b,$$
and $BC = c$. Q. E. F.

NOTE 1. If b and c were together not greater than a, then the two circles would not cut each other, and the construction would fail, which is also evident from Th. 11.

NOTE 2. The circles would also cut at a point on the other side of DF. Hence, with the given parts, *two* triangles can be constructed one on either side of DF.

PROBLEM 8.

To construct a triangle having given two sides and the included angle.



Let a, b be two given str. lines, and C a given angle.

It is required to construct a triangle having two of its sides equal to a and b, and having the angle included between these two sides equal to the ∠C.

Construction. Take any str. line AE.

At the point A in the str. line AE make the ∠EAF = the ∠C.

From AE cut off AB = a and from AF cut off AD = b.

Join BD.

Then ABD is the Δ required,
for, AB = a,
AD = b,
and the ∠BAD = the ∠C.

Q. E. F.

EXERCISES (*Numerical*)

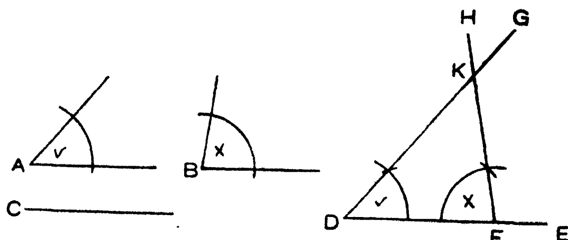
1. Construct a triangle two of whose sides are 5 *cms.* and 4 *cms.* and the included angle is 45°.

2. Construct a triangle two of whose sides are 6″ and 3″ and the included angle is 60°.

Verify that it is a rt. ∠ed triangle.

PROBLEM 9.

To construct a triangle having given two angles and the side adjacent to them.



Let A and B be two given ∠s, and C, a given str. line.

It is required to construct a Δ *having two of its* ∠*s equal to the* ∠*s* A *and* B, *and the side adjacent to these angles = the straight line* C.

Construction. Take any str. line DE, and from it cut off DF = C.

At the pt. D in the str. line DF, make the ∠FDG = the ∠A.
At the pt. F in the str. line FD, make the ∠DFH = the ∠B.

Let FH cut DG at K.

Then evidently, KDF is the Δ reqd.

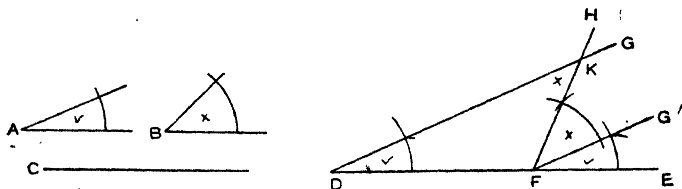
Q. E. F.

EXERCISES (*Numerical*)

1. Construct an equiangular triangle.
2. Construct a triangle ABC having given,
   (i) AB = 3″, ∠CAB = ∠CBA = 45° ;
   (ii) BC = 5 *cms*., ∠ABC = 60°, ∠ACB = 30°.

PROBLEM 9A.

To construct a triangle having given two angles and a side opposite to one of them.



Let A and B be the two given angles and C, a given str. line.

It is required to construct a triangle having two of its angles equal to the angles A *and* B, *and the side opposite to one of these angles, say, the angle which is equal to the given angle* B, *equal to the given line* C.

Construction. Take any str. line DE and from it cut off DF = C.

At the points D, F in DE make ∠^{s} EDG and EFG′ each equal to the ∠A.

At F on FG′, make ∠G′FH equal to the ∠B.

Let FH cut DG at K.

Then, KDF is the Δ reqd.

Proof. Because, the ext. ∠EFG′ = the int. opposite ∠EDG. *Cons.*

∴ FG′ is par^l to DG.

Hence, ∠DKF = the alt. ∠KFG′ = ∠B. *Cons.*

Hence, in the ΔKDF,

∠FDK = ∠A,

∠DKF = ∠B,

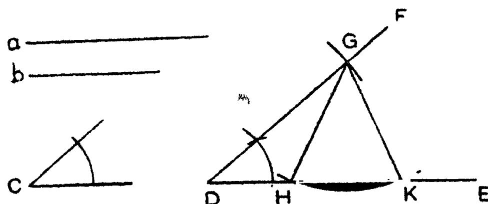
and the side DF = C.

∴ KDF is the Δ reqd.

Q. E. F.

PROBLEM 10.

To construct a triangle having given two sides and the angle opposite to one of them.



Let a, b be two given str. lines, and C the given angle.

It is required to construct a Δ *having two of its sides equal to a and b, and having the* ∠ *opposite to the latter side = the* ∠ C.

Construction. Take any str. line DE.

At the pt. D in the str. line DE, make the ∠EDF = the ∠C.

From DF cut off DG = a.

With centre G, and radius b, describe an arc of a ⊙ cutting DE at H and K.

If H and K are on the same side of D, as in the above diagram, join both GH and GK.

Then, evidently *each* of the Δ^s GDH and GDK is the Δ reqd.

Q. E. F.

NOTE 1. There are two solutions to the problem, as above, only when b is less than a but greater than the perpendicular from G on DE. This is known as the **Ambiguous Case.**

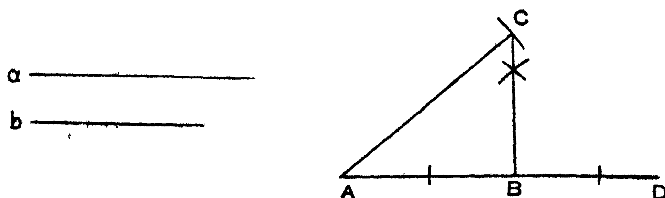
The cases when $b =$, or, $>$ a admit of one solution, each, which the student can easily verify.

NOTE 2. When b is less than p, the perpendicular from G on DE, the construction fails ; or, in other words, the problem does not admit of a solution.

Again, if $b = p$, there is only one triangle satisfying the given conditions.

PROBLEM 11.

To construct a right-angled triangle having given the hypotenuse and one side.



Let a be the given hypotenuse and b the given side.

Construction. Take any str. line AD, and from it cut off AB $= b$.

At B, construct the perpendicular BC upon AD.

With centre A and radius equal to a describe a circle cutting BC at C.

Join AC.

Then, evidently ABC is the Δ reqd.

Proof. By construction, AC $= a$,

AB $= b$,

and the $\angle$ ABC is a rt. angle.

NOTE. Evidently, there is another Δ on that side of AB remote from C.

Q. E. F.

ALTERNATIVE METHOD

Construction. Take any str. line AD cut off, from it, AB $= a$ the hypotenuse. Bisect AB at O and draw a $\odot$ with O as centre and OA or OB as radius. Again, with B as centre and b (the given side) as radius draw a $\odot$ to cut the former at C. Then the Δ ACB is the reqd. one.

EXERCISE 17

(On the Construction of Triangles)

1. Construct a triangle whose sides are

(i) 8 cm., 6 cm. and 4 cm.; (ii) 5 in., 2 in. and 4 in.;
(iii) 1·5 in.; 3 in.; 2·4 in.; (iv) 4 cm., 5·2 cm.; 3·3 cm.

2. Construct a triangle two of whose sides and the angle between them are, respectively,

(i) 5 cm., 2 cm.; 90°; (ii) 4 in., 3 in.; 45°;
(iii) 6 cm., 8 cm.; 135°; (iv) 2·5 cm.; 4·1 cm.; $\frac{1}{4}$ rt. angle.

3. Construct a triangle of which two angles and the side adjacent to them are

(i) 50°, 65°; 3 cm.; (ii) 90°, 30°; 6 cm.; (iii) 75°, 60°; 5 in.

4. Construct a triangle of which two angles and a side opposite to the first of these angles are

(i) 30°, 45°; 3 cm.; (ii) 60°, 90°; 2 in.; (iii) 75°, 62°; 3·5 in.

5. Construct a triangle of which two sides and the angle opposite to the first of these sides are

(i) 6 cm., 4 cm.; 120°; (ii) 8 cm., 7 cm.; 135°;
(iii) 3 in., 3·5 cm.; 60°; (iv) 3·2 in., 3·7 in.; 30°;
(v) 7 cm., 5 cm.; 45°.

6. Construct a right-angled triangle the hypotenuse and a side of which are

(i) 3·5 cm., 2 cm.; (ii) 4·7 in., 3·2 in.; (ii) 9 cm., 5 cm.

7. Take a straight line one inch in length, and construct an equilateral triangle on it. Hence construct angles of 60° and 30° at a given point.

8. ABC is a right angle. Through B draw a straight line BD within the angle so that the angle ABD may be equal to one-third of the angle ABC.

9. Trisect a right angle.

10. Take a straight line one and a half inches long, and on it construct an isosceles triangle having each of the two sides $2\frac{1}{4}$ inches in length.

11. AB is a given straight line. Show how to draw a straight line AC so that the angle BAC may be equal to 30°.

12. On a given straight line as hypotenuse, construct a right-angled triangle of which the acute angles are 30° and 60°.

13. Construct an isosceles triangle having given the vertical angle and the length of the perpendicular from the vertex to the base.

14. Construct a triangle of which the base angles shall be equal to two given angles, and the perpendicular from the vertex to the base equal to a given straight line.

15. Construct a triangle of which the two sides and the perpendicular from the vertex to the base are given.

16. Construct an equilateral triangle having given the perpendicular from the vertex to the base.

17. Construct a triangle having given the base, one of the angles at the base, and the sum of the sides.

18. Two sides of a triangle are respectively 3 inches and 2 inches in length, and the angle opposite to the latter side is equal to 30°; construct the triangle.

Also construct the triangle when the length of the latter side is $3\frac{1}{2}$ inches.

Show that in the former case there are two solutions, whilst in the latter there is only one.

19. If the side BA of triangle ABC be produced to D so that AD=AC, and if DC be joined, prove that the angle BDC is half the angle BAC. Hence show how to construct an acute angled triangle having given a base angle, the vertical angle, and the sum of the sides.

20. ABC is a triangle. If D be taken on CB produced such that BD=AB, and if E be taken on BC produced such that CE=AC, and if AD and AE be joined; prove that the angle ADE is half of the angle ABC, and the angle AED is half of the angle ACB. Hence show how to construct a triangle having given the *perimeter* and the base angles.

CHAPTER VIII

PARALLELOGRAMS

67. A quadrilateral whose opposite sides are parallel is called a **parallelogram.**

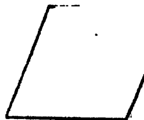


68. A parallelogram of which one angle is a right angle is called a **rectangle.**

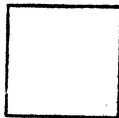


NOTE. It will be proved hereafter that if one angle of a parallelogram be a right angle *all* its angles are right angles.

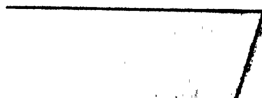
69. A quadrilateral whose sides are equal, but whose angles are not right angles is called a **rhombus.**



70. A rectangle of which two adjacent sides are equal, is called a **square.**

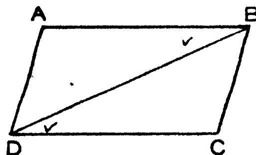


71. A quadrilateral which has one pair of opposite sides parallel, is called a **trapezium.**



THEOREM 22.

If a quadrilateral be such that two of its opposite sides are equal and parallel, then the other two sides also are equal and parallel.



Let ABCD be a quadrilateral such that two of its opposite sides AB, DC are equal and parallel.

It is required to prove that the other two sides AD, BC *are also equal and parallel.*

Proof. Join BD.

Then, because AB and DC are parallel, and BD meets them,

∴ the ∠ABD = the alt. ∠BDC.

Now in the △ˢ ABD, CDB, we have

AB = CD,

BD common,

and the ∠ABD = the ∠CDB; *proved above.*

∴ the two △ˢ are congruent.

Hence AD = CB; ....(i)

and the ∠ADB = the ∠CBD.

But these two are alt. ∠ˢ;

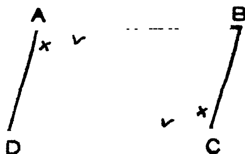
∴ AD and BC are parallel..........................(ii)

Thus, from (i) and (ii), AD and BC are both equal and parallel.

Q. E. D.

THEOREM 23.

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram.



Let ABCD be a parallelogram, of which AC is a diagonal.

It is required to prove that

(i) AB = DC ;
(ii) AD = BC ;
(iii) *the* ∠ ABC = *the* ∠ ADC ;
(iv) *the* ∠ BAD = *the* ∠ BCD ;
(v) *the* Δ ABC = *the* Δ ADC, *in area.*

Proof. Because AB, DC are parallel, and AC meets them.
∴ the ∠ BAC = the alt. ∠ ACD.

Also, because AD, BC are parallel, and AC meets them,
∴ the ∠ DAC = the alt. ∠ ACB.

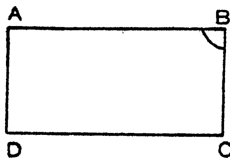
Hence, in the Δˢ ABC, ADC, we have
the ∠ BAC = the ∠ ACD,
the ∠ BCA = the ∠ CAD, } *proved above.*
and AC common ;
∴ the two Δˢ are congruent. *Th.* 18.

Hence AB = DC,
AD = BC,
the ∠ ABC = the ∠ ADC }
and the Δ ABC = the Δ ADC in area.

Also, since the $\angle$ BAC = the $\angle$ ACD, and the $\angle$ DAC = the $\angle$ ACB ; } *proved above.*

$\therefore$ the whole $\angle$ BAD = the whole $\angle$ BCD. Q. E. D.

COR. 1. *A rectangle has all its angles right angles.*



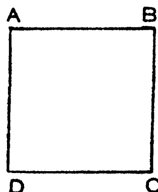
Let ABCD be a rectangle. Then dy definition, (i) it is a parallelogram and (ii) it has one angle, say the $\angle$ B, a right angle.

Since AB and DC are ||, the $\angle$ B + the $\angle$ C = 2 rt. $\angle$ ^s^; hence, the $\angle$ C *is a rt.* $\angle$.

Again, because ABCD is a par^m^, the $\angle$ B = the $\angle$ D, and the $\angle$ C = the $\angle$ A; hence the $\angle$ ^s^ A and D also are rt. $\angle$ ^s^.

$\therefore$ All the angles of the rectangle are right angles.

COR. 2. *A square has all its sides equal and all its angles right angles.*



Let ABCD be a square ; then by definition, (i) it is a rectangle and (ii) has two adjacent sides, say, AB and BC, equal.

Since, it is a rectangle, $\therefore$ *all its* $\angle$ ^s^ *are rt.* $\angle$ ^s^.

Again, since the square is also a par^m^ ;

$\therefore$ AB = DC and BC = AD.

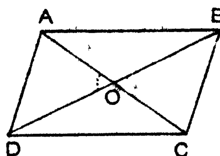
But AB = BC, by hypothesis.

Hence, all the sides of the square are equal.

COR. 3. *Parallel straight lines are always equidistant from one another.*

THEOREM 23A.

The diagonals of a parallelogram bisect one another.



Let the diagonals AC, BD of the par^{m} ABCD intersect at O.

It is required to prove that AO = OC, *and* BO = OD.

Proof. In the Δ^{s} AOD, COB, we have
the $\angle$OAD = the alt. $\angle$OCB,
the $\angle$ODA = the alt. $\angle$OBC,
and AD = BC ;

$\therefore$ the two Δ^{s} are congruent.

Hence, AO = OC, and BO = OD.

Q. E. D.

EXERCISE 18

1. If one of the angles of a parallelogram is (i) 60° ; (ii) 135° ; (iii) 150°, write down its other angles.

2. If in the figure of Th. 33, $\angle$BAC = 30° and $\angle$DAC = 45°, write down the values of all the other angles in the figure.

3. A quadrilateral is a parallelogram (i) if its opposite sides are equal ; or (ii) if its opposite angles are equal or, (iii) if the diagonals of a quadrilateral bisect each other.

4. ABCD is a parallelogram ; if E and F are the middle points of AB and CD respectively, prove that the quadrilateral AECF is a parallelogram.

5. The bisectors of two adjacent angles of a parallelogram meet at right angles.

6. ABCD is a parallelogram. If CD is bisected at O and the $\angle$AOB is a rt. $\angle$, prove that AB = 2BC.

7. If the diagonals of a parallelogram are equal, prove that it is a rectangle.

8. From any two points on a straight line, perpendiculars are drawn to a parallel straight line; prove that these perpendiculars are equal.

9. ABC and A′B′C′ are two triangles such that AB, AC are respectively equal and parallel to A′B′, A′C′. Prove that BC also is equal and parallel to B′C′.

10. Prove that the diagonals of a rhombus bisect each other at right angles.

11. If in a trapezium the non-parallel sides are equal, prove that the angles adjacent to either of the parallel sides are equal.

12. If the sum of the distances of any angular point of a quadrilateral from the other three is the same for all four, the figure is a rectangle.

13. If from any point on the diagonal of a rhombus lines are drawn parallel to the sides, show that each of the two parallelograms so formed about the diagonal of the rhombus is also a rhombus.

14. The diagonals of a quadrilateral are equal and bisect one another at right angles. Prove that the quadrilateral is a square.

15. The diagonal AC bisects the $\angle$BCD of the par^{m} ABCD. Prove that the par^{m} is a rhombus.

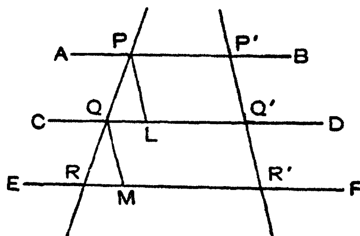
16. Any str. line drawn through the point of intersection of the diagonals of a parallelogram and terminated by a pair of opposite sides of the par^{m} is bisected at that point.

17. The str. lines bisecting two opposite angles of a par^{m} are either parallel or coincident.

18. The bisectors of the angles of a par^{m} whose sides are not all equal from a rectangle.

THEOREM 24.

If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.



Let AB, CD, EF be parallel str. lines. Let a str. line PR cut them in P, Q, R respectively so that the intercepts PQ, QR are equal.

Let any other str. line P′Q′ cut the par^l lines in P′, Q′, R′.

It is required to prove that P′Q′ = Q′R′.

Suppose PL, QM are drawn par^l to P′R′, meeting CD in L and EF in M.

Proof. Since CD is ∥ to EF, and PR meets them.
∴ the ∠PQL = the corresponding ∠QRM.

Also, PL and QM are parallel, each being ∥ to P′R′;
∴ the ∠QPL = the corresponding ∠RQM.

Then, in the Δ^s PQL and QRM,
the ∠PQL = the ∠QRM,
the ∠QPL = the ∠RQM,
and PQ = QR;
∴ the two Δ^s are congruent.
Hence PL = QM.

Now, the quadrilateral PLQ′P′ is a par^{m}, since its opp. sides are ∥.

∴ PL = the opposite side P′Q′.

Similarly, the quadl. QMR′Q′ being a par^{m},

QM = Q′R′.

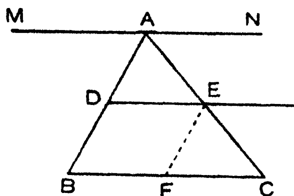
But, PL = QM, *proved*

∴ P′Q′ = Q′R′.

Q. E. D.

COR. 1. *If* D *be the middle point of the side* AB *of a triangle* ABC, *then the straight line drawn through* D *parallel to* BC *will bisect* AC.

Let the str. line through D ∥ to BC intersect AC at E.



It is required to prove that AE = EC. If through A a str. line MN be supposed to be drawn ∥ to BC, then the parallels MN, DE, BC are cut by the two str. lines AB, AC.

Hence, the intercepts made on AB being equal, the corresponding intercepts on AC must be equal, too.

Hence, AE = EC.

COR. 2. *If* D *and* E *be the middle points of the sides* AB *and* AC *of a triangle* ABC, *then* DE *is parallel to* BC *and is half of* BC.

For, if DE be not ∥ to BC, let DE′ be ∥ to it, meeting AC in E′. Then E′ is the mid-point of AC, which is impossible. Hence DE is ∥ to BC.

Again, it is required to prove that DE = $\frac{1}{2}$.BC.

Draw EF par^{l} to AB meeting BC at F.

Since E is the mid-point of AC and EF is par' to AB,

$\therefore$ BF = FC *Cor. 1.*

i.e., BF = $\frac{1}{2}$.BC.

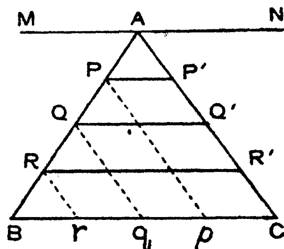
But, BAEF is a par^m, its opposite sides being par'.

$\therefore$ BF = the opposite side DE.

Hence, OE = $\frac{1}{2}$.BC.

COR. 3. *If one side of a triangle be divided into any number of equal parts, and if through the points of division, straight lines be drawn parallel to the base, then the points in which these parallels meet the other side will divide that side into the same number of equal parts.*

If the points P, Q, R divide the side AB of a triangle ABC into *four* equal parts; and if PP', QQ', RR' drawn ∥ to BC meet AC in P', Q', R', then AC also is divided into *four* equal parts at these points.



Supposing MAN to be drawn ∥ to BC, we find that the five parallels make equal intercepts on the line AB which cuts them.

Hence the intercepts made by them on another line AC, which cuts them, must also be equal.

If through P, Q, R lines Pp, Qq, Rr be drawn ∥ to AC, they will also divide BC into four equal parts, each of which is equal to PP'. Further, it is evident that QQ' = 2PP', RR' = 3PP', *i.e.*

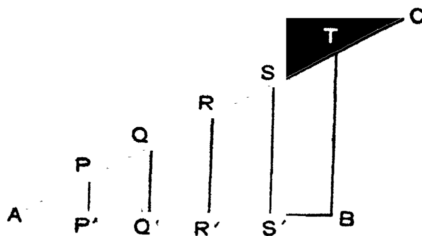
PP' = $\frac{1}{4}$.BC, QQ' = $\frac{2}{4}$.BC, RR' = $\frac{3}{4}$.BC.

Similarly, if AB is divided into n equal parts by the par' lines,

PP' = $\frac{1}{n}$. BC, QQ' = $\frac{2}{n}$. BC; and so on.

PROBLEM 12.

To divide a given straight line into any number of equal parts (say five).



Let AB be the given str. line.

It is required to divide it into five equal parts.

Construction. From A draw a str. line AC, of unlimited length, making any angle with AB.

From AC cut off successively five *equal* parts of *any* length, AP, PQ, QR, RS, ST.

Join TB ; and through P, Q, R, S draw PP′, QQ′, RR′, SS′ each ∥ to TB, meeting AB in P′, Q′, R′, S′ respectively.

Then AB is divided into five equal parts at the points P′, Q′, R′, S′.

Proof. Since the side AT of the Δ ATB has been divided into five equal parts at the points P, Q, R, S, and through the points of division PP′, QQ′, RR′, SS′ have been drawn ∥ to the base TB ;

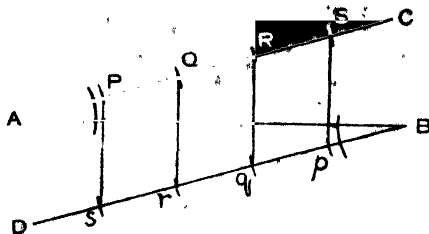
∴ the points P′, Q′, R′, S′ also divide the other side AB into five equal parts.

Q. E. F.

NOTE. In the above figure, AP′ = $\frac{1}{5}$.AB, AQ′ = $\frac{2}{5}$.AB, etc. Hence, the above construction may also be used to cut off a given fractional part from a line.

PROBLEM 12. SECOND METHOD.

To divide AB *into five equal parts.*



Construction. Through A, draw the str. line AC at any angle with AB. Through B, draw BD par^l to AC.

From AC, cut off equal parts AP, PQ, QR, RS.

Also, from BD cut off parts Bp, pq, qr, rs equal to the parts taken on AC.

Join Sp, Rq, Qr, Ps.

These lines divide AB into five equal parts.

Because RS and pq are equal and parallel by construction.

∴ Rq is par^l to Sp. *Th.* 22.

Similarly, Qr is par^l to Rq and Ps is par^l to Qr.

∴ Ps, Qr, Rq and Sp are par^l to one another.

Next apply Th. 24, Cor. 3, for the proof.

EXERCISE 19

1. *The str. line drawn through the middle point of a side of a triangle parallel to the base bisects the remaining side.* Prove this without applying Th. 24.

[In the ΔABC, if D be the middle point of AB and DE is drawn par^l to BC meeting AC in E, it is required to prove that AE=EC.

Through E draw EF par^l to AB meeting BC in F, and then prove that the Δs ADE and EFC are congruent.]

2. *The str. line joining the middle points of two sides of a triangle is parallel to and half of the third side.* Prove this without applying the indirect method.

[In the ΔABC, let D, E be the mid-pts. of the sides AB, AC. To prove that DE is par[l] to BC and $= \frac{1}{2}$.BC.

Produce DE to F such that EF=DE and join CF. Then, prove that the Δs ADE, ECF are congruent and the quadrilateral BDFC is a par[m].]

3. If the middle points of the sides of a triangle be joined, prove that the triangle is divided into four congruent triangles.

4. If through the middle point of a side of a triangle a str. line be drawn parallel to the base, prove that it bisects any str. line drawn from the vertex to the base.

5. D is the mid-point of the hypotenuse BC of the right-angled triangle ABC. Prove that BC=2AD. [Join D with mid-point of AB.]

6. If the middle points of the adjacent sides of any quadrilateral are joined, prove that the figure thus formed is a parallelogram.

7. Straight lines joining the middle points of the opposite sides of a quadrilateral bisect each other.

8. The str. lines joining the middle points of opposite sides of a quadrilateral, and the middle points of the diagonals, meet in a point, which is the middle point of all the three lines.

9. The str. line joining the middle points of the oblique sides of a trapezium is parallel to the parallel sides and is equal to half their sum.

10. C is the middle point of the str. line AB. If Aa, Bb, Cc be drawn perpendiculars from A, B, C on the str. line acb, prove that c is the mid-point of ab.

11. Show how to construct a triangle given the three middle points of its sides.

12. Divide a str. line into (i) 9 equal parts; (ii) 11 equal parts; (iii) 7 equal parts.

13. From a str. line cut off a portion equal to (i) $\frac{4}{5}$ of the line; (ii) $\frac{3}{7}$ of the line; (iii) $\frac{5}{11}$ of the line.

14. Construct an equilateral triangle whose perimeter is given.

15. Divide a given straight line into two parts one of which is (i) $\frac{5}{4}$ of the other; (ii) $\frac{2}{5}$ of the other; (iii) $\frac{3}{8}$ of the other.

16. Divide a given str. line into 3 parts such that the first part is double of the second and half of the third.

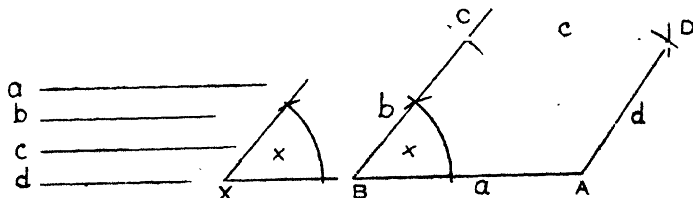
CHAPTER IX

THE CONSTRUCTION OF QUADRILATERALS

72. We have noticed that three independent data are required for the construction of a triangle; but it will be evident from the following examples that *five* independent data are necessary to construct a quadrilateral.

PROBLEM 13.

To construct a quadrilateral having given the lengths of the four sides and one angle.



Let a, b, c, d be the lengths of the four sides and X the angle between the sides, equal to a and b.

Construction. Take a str. line AB and cut off from it a portion AB $= a$.

Make the $\angle$ABC $=$ X and from BC cut off BC $= b$. With C and A as centres and radii equal to c and d respectively draw two arcs intersecting at the point D.

Join DC, DA.

Then, ABCD is the quadrilateral reqd.

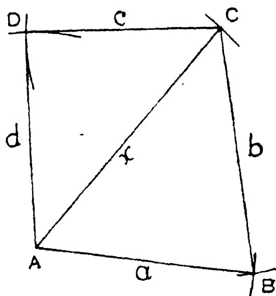
Proof. By construction, the sides of ABCD are equal to a, b, c, d, and the $\angle$ABC $=$ X.

Q. E. F.

PROBLEM 14.

To construct a quadrilateral having given the four sides and a diagonal.

a
b
c
d
x



Let a, b, c, d be the lengths of the four sides and x the length of the diagonal.

Construction. Take a line AC $= x$.

With A and C as centres and radii equal to a and b respectively draw two arcs intersecting at B.

Again, with A and C as centres and radii equal to d and c draw two other arcs intersecting at D.

Join BA, BC and DA, DC.

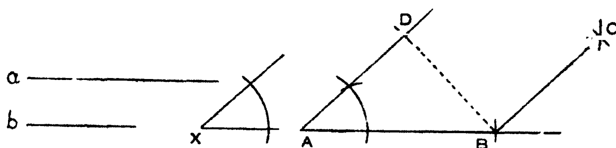
Then, ABCD is the quadrilateral reqd.

Proof. By construction, the sides of the quadrilateral ABCD are equal to a, b, c, d and the diagonal AC $= x$.

Q. E. F.

PROBLEM 15.

To construct a parallelogram having given two adjacent sides and the included angle.



Let a and b be the two given sides, and X the given angle.

Construction. Take a line AB and from it cut off AB $= a$.

Make the angle BAD = X and from AD cut off AD = b.

With centres B and D, and radii equal to b, a respectively draw two arcs intersecting at C.

Join CB, CD.

Then ABCD is the par^{m} reqd.

Proof. Join DB.

In the Δ^{s} ABD, DBC,

AB = CD

AD = BC

and BD is common.

∴ the two Δ^{s} are congruent;

so that, the ∠ABD = the ∠CDB

and the ∠ADB = the ∠DBC.

But these are pairs of alternate angles.

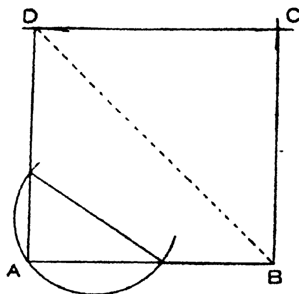
∴ AB is ∥ to CD and AD is ∥ to BC

i.e., ABCD is a par^{m}.

Q. E. F.

PROBLEM 16.

To construct a square on a given side.



Let AB be the given side.

Construction. At A draw AD perpendicular to AB and cut off from it AD = AB.

With B and D as centres and with radius AB, draw two arcs cutting at C.

Join CB, CD.

Then ABCD is the square required.

Proof. Join DB. As in Prob. 15, ABCD may be shewn to be a par^{m}.

Because the ∠BAD is a rt. ∠ and also by construction the sides of the figure are equal.

∴ ABCD is a square.

Q. E. F.

EXERCISE 20

(On Problems 13-16).

1. Construct a quadrilateral having given
   (i) four sides and an angle ,
   (ii) three sides and two diagonals ;
   (iii) three sides and the two angles between them ;
   (iv) the two segments of each diagonal and the angle between them ;
   (v) two adjacent sides and three angles.

2. Construct a parallelogram having given
   (i) two adjacent sides and a diagonal ;
   (ii) two adjacent sides and the angle between a diagonal and one of the two sides ;
   (iii) a side and two diagonals ;
   and (iv) two diagonals and the included angle.

3. Construct a trapezium having given the parallel sides, and the other two.

[If ABCD is the trapezium with AB ∥ to CD, through B draw BE ∥ to AD to meet CD at E. Then, clearly the sides of the △CEB are known and ABED is also a known par^{m}.]

4. Construct a rhombus having given
   (i) a side and an angle ;
   (ii) a side and a diagonal ;
   and (iii) two diagonals.

5. Construct a rectangle having given
   (i) two adjacent sides ;
   (ii) a diagonal and a side ;
   and (iii) a diagonal and the angle between the diagonals.

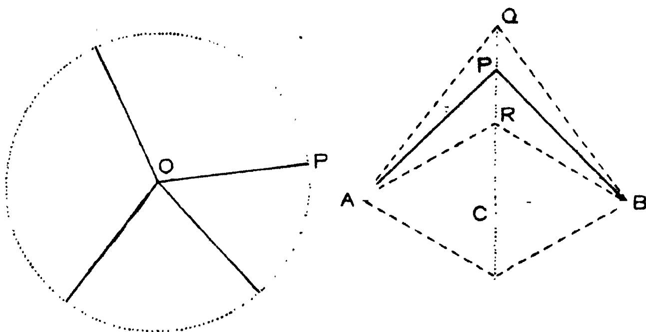
6. Construct a square having given a diagonal.

[Diagonals of a square are equal and bisect each other at rt. angles.]

CHAPTER X

LOCI

73. When a point moves and its motion is restricted by some given condition, the path traced out by the point is called the **locus.**



For instance, when a point P moves so that its distance from a fixed point O, is always the same, say, 2 cms., its locus is evidently the circumference of a circle of which the fixed point O is the centre and the given distance (*viz.* 2 cms.), the radius.

When any number of points, each of which satisfies a given condition, lie on a particular line (straight or curved), that line is also called the locus of those *points.* From this point of view, the locus of points, each of which lies at a given distance from a given point, is the circumference of the circle which is described with the given point as centre and the given distance as radius.

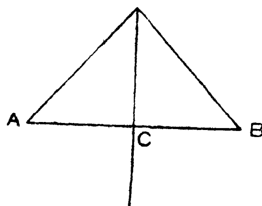
74. In order to show that the locus of a point is a certain line (straight or curved) it must be proved that (i) *every point which satisfies the given condition lies on the line*;

and (ii) *every point on the line satisfies the given condition.*

In many cases, however, when the first part is proved, the second part becomes evident, and is not, therefore, established.

THEOREM 25.

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.



Let A and B be two fixed points.

It is required to prove that the locus of a point equidistant from A *and* B *is the perpendicular bisector of* AB.

Cf. Fig. Art. 73.

i.e. to prove that (*i*) *if any point is equidistant from* A *and* B, *it is on the perpendicular bisector of* AB ;

and (ii) *if any point lies on the perpendicular bisector of* AB, *it is equidistant from* A *and* B.

(i) Let P be any point equidistant from A and B so that PA = PB.

It is required to prove that P *lies on the perpendicular bisector of* AB.

Proof. Let C = the mid-pt. of AB.

Join PC.

In the Δ^s PCA, PCB,

PA = PB

CA = CB

and PC is common.

$\therefore$ The two Δ^s are congruent.

Hence, the $\angle$ACP = $\angle$BCP.

$\therefore$ CP is perp. to AB,

i.e. P lies on the perpendicular bisector of AB.

(ii) Let P be any point on the perpendicular bisector PC of AB.

It is required to prove that P *is equidistant from* A *and* B.

Proof. In the Δ^s PCA, PCB,

AC = BC,

PC is common,

and the $\angle$PCA = the $\angle$PCB, each being a rt. $\angle$.

$\therefore$ The two Δ^s are congruent.

Hence, PA = PB.

i.e., P is equidistant from A and B.

$\therefore$ From (i) and (ii), the required locus of P is the perpendicular bisector of AB.

Q. E. D.

EXERCISE 20A

1. Prove that the locus of a point which moves in such a manner that its distance from a given straight line is always the same, consists of two straight lines parallel to the given line, one on either side of it.

2. Find the locus of the vertex of an isosceles triangle on a given base AB.

3. Str. lines are drawn from the vertex A to the base BC of the ΔABC. Find the locus of their middle points.

4. The locus of the vertices of right-angled triangles described on a given hypotenuse AB, is the circle on AB as diameter.

5. A straight rod PQ of fixed length slips between two fixed rods OA, OB at rt. angles of each other. The locus of the middle point of PQ is a circle with centre O and radius = $\frac{1}{2}$.PQ.

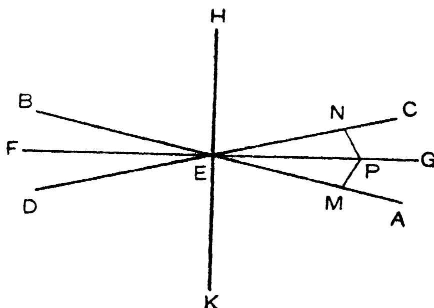
6. Find the locus of the centre of circles passing through two given points.

7. The locus of a point which is equidistant from two given par' lines is the line par' to each of them and lying half way between them.

8. Find a point in a given straight line at a given distance from another given straight line.

THEOREM 26.

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.



Let AB, CD be two given str. lines which intersect at E.

It is required to prove that the locus of a point equidistant from AB *and* CD *is the pair of lines which bisect the angles between* AB *and* CD.

i.e., to prove that

(i) *if any point is equidistant from* AD *and* CD. *it lies on either of the bisectors of the angles between* AB *and* CD;

and (ii) *if any point lies on any of the bisectors of the angles between* AB *and* CD, *it is equidistant from* AB *and* CD.

(i) Let P be any point equidistant from AB and CD so that the perp. PM on AB = the perp. PN on CD.

It is required to prove that the ∠PEM = *the* ∠PEN.

Proof. In the right-angled Δ^s PEM, PEN,
the hypotenuse PE is common
and PM = PN;
∴ The two Δ^s are congruent.
Hence, ∠PEM = ∠PEN.

(ii) Next, let P be any point on the bisector of any one of the angles between AB and CD.

It is required to prove that P *is equidistant from* AB *and* CD.

Proof. From P draw PM and PN ⊥ to AB and CD respectively.

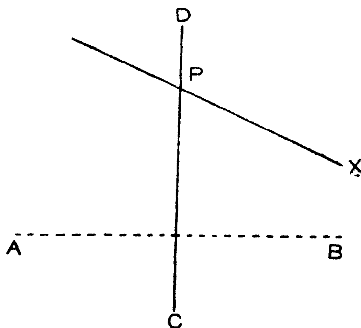
In the Δ[s] PEM, PEN, we have
the ∠PEM = the ∠PEN, *Hyp.*
the ∠PME = the ∠PNE, *Rt.* ∠*s.*
and the side EP common;
∴ the two Δ[s] are equal in all respects.
Hence PM = PN.
i.e. P is equidistant from AB and CD.

∴ From (i) and (ii), the locus of a point equidistant from AB and CD is the pair of lines which bisect the angles between AB and CD. Q. E. D.

Intersection of Loci

75. When a point satisfies more than one geometrical condition it can be found out conveniently by the method of loci, as in the following illustrations.

Ex. 1. Find the point in a given str. line X so that its distances from two given points A and B outside the line X may be equal.

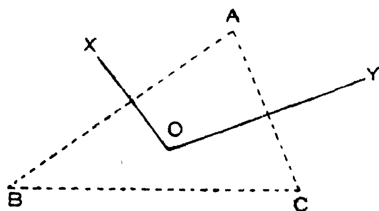


Since the locus of all the points equidistant from A and B is the perpendicular bisector, CD, of the line AB,

∴ the required point must lie on CD.
But it is also on X.

Hence, the reqd. pt. is common to both CD and X *i.e.* the point of their intersection, *viz.* P.

Ex. 2. Find a point equidistant from three given points A, B, C.

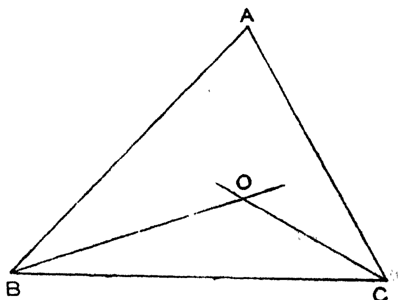


Since the point required is equidistant from A and B, it must be on the perpendicular bisector, X, of the line AB.

Again, since the point is also equidistant from A and C it must be on the perpendicular bisector, Y, of the line AC.

Hence, the point required must be common to both X and Y *i.e.* the point of intersection O of the lines X and Y.

Ex. 3. Find a point equidistant from the sides BC, CA, AB of the △ABC.



The locus of points equidistant from BC, AB is the bisector, BO of the ∠ABC.

Similarly, the locus of points equidistant from BC, CA is the bisector, CO of the ∠BCA.

Hence, the point common to BO and CO must satisfy both these conditions, *i.e.* O, the point of intersection of BO and CO must be equidistant from BC, CA, AB.

EXERCISE 21

1. What is the locus of points which are equidistant from two given points? Construct an isosceles triangle on a given base AB and having its vertex on a given line.

2. What is the locus of points which are equally distant from two intersecting straight lines?

Find a point in a given straight line so that it may be equally distant from two intersecting straight lines.

3. Find a point which is equally distant from two fixed points and is also at a given distance from a third fixed point.

When is this problem impossible?

4. Find a point which is equally distant from two fixed straight lines and is also at a given distance from a fixed point.

5. Find a point which is equally distant from two fixed straight lines and from two fixed points.

6. Find a point which is at a distance of 5 cms. from two fixed intersecting lines.

7. Find the centre of a circle passing through three given points.

8. Draw a circle passing through two given points and having a given radius.

9. Find a point which is at a distance of 2 inches from one fixed point and 3 inches from another.

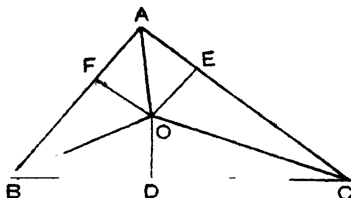
When is this problem impossible?

10. AB and AC are two fixed and unlimited straight lines. Find a point which is at a distance of 2 inches from AB and 3 inches from AC.

CHAPTER XI

MISCELLANEOUS PROPOSITIONS (I)

1. *The bisectors of the angles of a triangle meet at a point.*



Let ABC be a $\triangle$, and let the bisectors of the $\angle^s$ B and C meet at O.

Join AO

It is required to prove that AO *is the bisector of the* $\angle$ A.

From O draw OD, OE, OF $\perp$ to BC, CA, AB respectively.

Proof. Because BO bisects the $\angle$ABC,

$\therefore$ any pt. in BO is equidistant from BC and BA;

$\therefore$ OD = OF.

For a similar reason, any pt. in CO is equidistant from CB and CA;

$\therefore$ OD = OE.

Hence OF = OE.

$\therefore$ O is on the bisector of the $\angle$BAC;

i.e., AO is on the bisector of the $\angle$BAC.

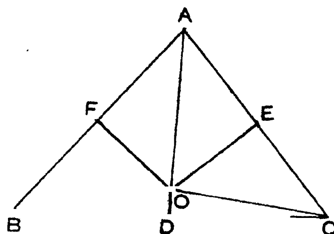
Thus, the bisectors of the $\angle^s$ A, B, C meet at a pt. Q. E. D.

76. When three or more straight lines meet at a point, they are said to be **concurrent.**

So the above proposition might be stated as follows:—*The bisectors of the angles of a triangle are concurrent.*

When three or more points lie in the same straight line they are said to be **collinear.**

2. *The perpendiculars drawn to the sides of a triangle from their middle points are concurrent.*



Let ABC be a △, and D, E, F be the mid-points of BC, CA, AB respectively.

From F and E draw ⊥ˢ to AB, AC meeting at O.

Join OD.

It is required to prove that OD *is* ⊥ *to* BC.

Join OA, OB, OC.

Proof. Because O is on the perpendicular bisector of AB.

∴ OA = OB.

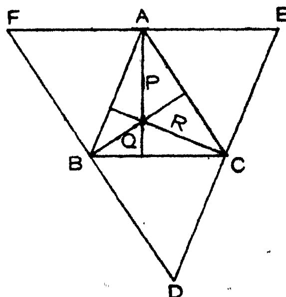
For a similar reason, OA = OC.

Hence OB = OC;

and ∴ O is on the ⊥ bisector of BC.

Hence OD is ⊥ to BC. Q. E. D.

3. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*



Let ABC be a △, and let AP, BQ, CR, when produced, be ⊥ to BC, CA, AB respectively.

It is required to prove that AP, BQ, CR (*when produced*) *are concurrent.*

Through A, B and C, draw str. lines ∥ to BC, CA and AB respectively, thus forming the △DEF.

Proof. Since FE is ∥ to BC, and FD is ∥ to AC,

∴ AFBC is a par^m.

∴ FA = BC.

Similarly, AE = BC.

Hence, FA = AE; and ∴ A is the mid-point of EF.

Similarly, B is the mid-pt. of FD, and C is the mid-pt. of DE.

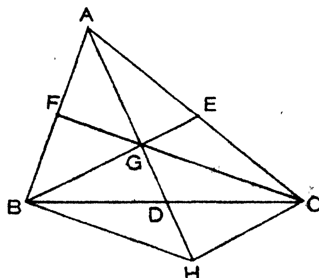
Now, since BC is ∥ to FE, and AP produced is ⊥ to BC,

∴ AP is also ⊥ to FE.

Similarly, BQ is ⊥ to FD and CR is ⊥ to DE.

Thus, AP, BQ and CR are ⊥ to the sides of the △EFD at their middle points; and ∴ they are concurrent. Q. E. D.

4. *The medians of a triangle are concurrent; the common point divides each median into two parts, of which the part nearer the vertex is double the other.*



Let E, F be the mid-pts. of the sides CA and AB of the △ABC, and let the medians BE, CF intersect at G.

Join AG, and produce it to meet BC at D.

It is required to prove that D *is the mid-pt. of* BC.

Through B draw BH ∥ to FC, and produce AD to meet BH at H.

Join CH.

Proof. (1) In the Δ ABH, F is the mid-pt. of AB, and FG is ∥ to BH;

∴ G is the mid-pt. of AH.

Again, in the Δ AHC, G and E are the mid-pts. of the sides AH and AC.

∴ GE is ∥ to HC.

Thus the opp. sides of the quadl. BGCH are ∥, and ∴ BGCH is a par^m.

But the diagonals of a par^m bisect one another;

∴ D is the mid-pt. of BC.

Thus the three medians AD, BE, CF meet at the point G.

Q. E. D.

(2) Since D is the mid-pt. of GH.

∴ GH = 2GD.

But AG = GH, ∴ AG = 2GD.

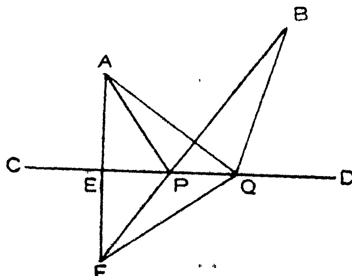
Also, BG = HC = 2GE; and, CG = HB = 2GF. *Th. 24, Cor. 2.*

Q. E. D.

77. The point of intersection of the medians of a triangle is called the **centroid** of the triangle.

Thus, in the above figure, the point G is the *Centroid* of the Δ ABC.

5. A *and* B *are two points on the same side of an unlimited straight line* CD. P *is a point in* CD *such that the angles* APC *and* BPD *are equal. Prove that, of all points on the straight line* CD, P *is that the sum of whose distances from* A *and* B *is the least.*



Let Q be *any other* pt. in CD.

Join AQ, BQ.

It is required to prove that AQ + QB > AP + PB.

Produce BP to F, making PF = PA; join AF, cutting CD in E.
Join FQ.

Proof. In the Δ' APE, FPE, we have

AP = PF,

PE common,

and the ∠APE = the ∠EPF

(∵ each of them = the ∠BPD).

∴ the two Δ' are congruent.

Hence, AE = EF, and EP is ⊥ to AF.

Thus, CD is the ⊥ bisector of the str. line AF.

Hence, QA = QF;

and ∴ AQ + QB = FQ + QB.

Also AP + PB = FB. *Cons.*

But FQ + QB is > FB,

∴ AQ + QB is > AP + PB. Q. E. D.

MISCELLANEOUS EXERCISES

1. A straight line AB is bisected at C. If any point P be taken on AB produced, prove that AP + BP = 2CP.

2. A straight line AB is bisected at C and any point P is taken in BC, show that AP − BP = 2CP.

3. Define adjacent angles.

POA, POB are angles on the same side of OP. If OC bisects ∠AOB, prove that ∠POA + ∠POB = 2∠POC.

4. AOP, BOP are adjacent angles, of which AOP is the greater. If OC bisects ∠AOB, prove that ∠AOP − ∠BOP = 2∠COP.

5. If, in the diagram of Th. 1, ∠AOC is one-third of ∠BOC, find ∠COD.

6. If the straight line joining the middle points of two opposite sides of a quadrilateral be perpendicular to each of these sides, prove that its other sides are equal.

7. If the bisector of the vertical angle of a triangle also bisects the base, prove that the triangle is either isosceles or obtuse-angled

8. The equal sides AB, AC of an isosceles triangle ABC are produced to D and E respectively such that BD = CE. Prove that BE = CD.

9. BC is the greatest side of the $\triangle$ABC, and E, F are any points in CA, AB respectively, prove that BC > EF.

10. No straight line can be drawn within a triangle greater than the greatest side.

11. In any triangle any two sides are together greater than twice the median which bisects the remaining side.

Hence deduce that *in any triangle the sum of the medians is less than the perimeter.*

12. *The diameter is the greatest chord of a circle.* [The straight line joining any two points on the circumference of a circle is called a *chord.*]

13. The angle made by two straight lines joining any point on a circle to the ends of a diameter is a rt. angle.

14. The angle between the bisector of the vertical angle of a triangle and the perpendicular from the vertex to the base is equal to half the difference of the base angles.

15. ABC is an equilateral triangle, and D is a point within it. Show that it is possible to construct a triangle with its sides equal to DA, DB, DC.

16. ABCD is a quadrilateral such that the bisectors of $\angle$s ABC, BCD make a rt. angle. Prove that AB is parallel to CD.

17. In a certain triangle the three angles are in the ratio 4 : 3 : 2. Find the number of degrees in each angle.

18. In a right-angled triangle, one acute angle exceeds the other by 30°. Find the angles.

19. How many sides has a regular polygon whose interior angle is 19 times as great as its exterior angle?

20. If the sum of the interior angles of a polygon is 24 rt. angles, find the number of its sides.

21. If each interior angle of a regular polygon be $1\frac{17}{21}$ rt. angles, find the number of its sides.

22. ABCDE is a regular pentagon, and AC is joined. Prove that the bisector of the $\angle$ACB is at right angles to CD.

23. Prove that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.

24. If a quadrilateral has all its sides equal, prove that its diagonals bisect each other at right angles.

25. ABC is a triangle of which the sides AB is greater than the side AC, if D is the middle point of BC and AD is joined, prove that the ∠ADB is obtuse.

26. If, in Ex. 25, E be any point on AD, prove that BE is greater than CE.

27. Any point P is taken within a circle with centre O and straight lines PA, PB are drawn to the circumference. If ∠POA > ∠POB, prove that PA > PB and conversely.

28. Through any point X within a circle the diameter AB is drawn such that the centre of the circle lies on AX. Prove that AX and BX are respectively the greatest and the least of all the straight lines that can be drawn from X to the circumference.

29. In the △ABC, the sides AB, AC are produced to D and E respectively such that BD = CE. If CD > BE, then AB > AC and *conversely*, if AB > AC, then CD > BE.

30. ABCD is a quadrilateral in which AB = CD and AC > BD. Prove that ∠ABC > ∠BCD.

31. ABC is a triangle of which the side AC is greater than the side AB. AD is drawn bisecting the angle BAC, and meeting BC in D. Prove that CD is greater than BD.

32. Through a given point draw a straight line such that the perpendiculars on it from two fixed points may be on opposite side of it and equal to each other.

33. ABCD is a parallelogram ; E and F are the mid-points of AB and CD respectively. If DE, BF intersect the diagonal AC at G and H, prove that AG = GH = HC.

34. Prove that the medians of a triangle meet at a point which is a point of trisection on each of them. Hence find a method of trisecting a given straight line.

35. Construct a triangle having given the base, the perpendicular from the vertex to the base, and the line joining the vertex to the mid-point of the base.

36. ABC is a triangle. If AB, AC be produced to D and E respectively, prove that the bisectors of the angle DBC, ECB meet at a point which lies on the bisector of the angle BAC.

37. Construct a right-angled triangle having given the hypotenuse and the sum of the sides.

38. Prove that the locus of the vertices of all right-angled triangles which have a common hypotenuse is a circle.

39. Construct an isosceles triangle having given the base, and the sum of one of the equal sides and the perpendicular from the vertex to the base.

40. AB, AC are two given straight lines; through a given pt. D between them, draw a straight line meeting AB, AC in G and H, so that GH may be bisected at D.

41. OA, OB, OC are three given straight lines, of which OB lies between OA and OC. Draw the straight line terminated by OA and OC, and bisected by OB.

42. ABCD is a parallelogram, and MN is an unlimited straight line outside it. If AP, BQ, CR, DS be perpendiculars to MN, prove that AP+CR=BQ+DS.

43. Construct a triangle, having given the base, an angle at the base, and the difference of the sides.

44. BC is the base of an isosceles triangle ABC, EAD is drawn parallel to BC, and *any* point P is taken on AD. Prove that BP+PC > BA+AC.

Prove also that if Q be *any* point above the line DE, BQ+QC >BA+AC.

Hence prove that if R be *any* point *above* BC, such that BR+RC=BA+AC, then R must be *below* DE.

45. Of all taiangles which stand on the same base and which have their vertices in a straight line parallel to the base, prove that the triangle which is isosceles, has the *least* perimeter.

46. Prove that any side of a triangle is greater than the difference of the other two sides. Hence prove that if P is a point in an unlimited straight line MN, such that PA and PB are equally inclined to it, where A and B are two fixed points on opposite sides of MN, then of all points in the given straight line, P is that the *difference* of whose distances from A and B is the *greatest*.

47. Construct a triangle having given the three medians.

[To the figure of Misc. Prop. No. 4; notice that the ΔBGH can be constructed.]

48. If two of the medians of a triangle are equal, the triangle is isosceles.

49. Find the locus of a point which moves so that the sum or difference of its distances from two intersecting str. lines at rt. angles to each other is always a constant.

50 If the sides of a triangle are unequal prove that the bisector of the vertical angle lies between the median and the perpendicular from the vertex to the base.

BOOK II

AREAS

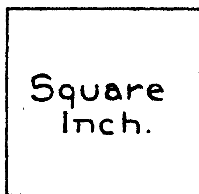
CHAPTER I

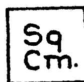
DEFINITIONS

1. The **area** of a figure is the amount of surface enclosed by its bounding lines.

2. A **square inch** is the area of a square described on a side one inch long.

A *square centimetre* is the area of a square described on a side one centimetre long.





Similarly, the terms a **square foot,** a **square yard,** a **square mile,** etc. are used in the same sense.

3. The **unit of area** is the area of a square on a side of one unit length.

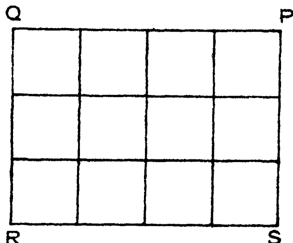
Thus, if the unit of length is an *inch*, the unit of area is a *square inch* ; if the unit of length is a *centimetre*, the unit of area is a square centimetre ; etc.

4. To **measure** the area of a figure is to find how often the figure contains the unit of area.

5. The area of a rectangle.

Let PQRS be a rectangle of which the length $PQ=4$ units and the breadth $QR=3$ units.

Divide PQ into four equal parts and QR into 3 equal parts and through the pts. of division in either side, draw lines parallel to the other; then the whole rectangle is divided into 3 rows of squares, a side of each square being equal to the unit of length. There are 4 squares in each row; the total number of squares = 3 times 4 = 12, and hence the area of the rectangle = 12 units of area.



Similarly, if the length = a units and the breadth = b units, the area of the rectangle = $a \times b$ units of area.

Hence, if the side of a square = a units,

$$\text{its area} = a \times a \text{ units of area}$$
$$= a^2 \text{ units of area,}$$

or, briefly, the **area of a rectangle = length × breadth ;**
the **area of a square = side × side.**

Notation

6. The square on the side AB is written as AB^2, or *sq. on* AB.

A rectangle is said to be **contained** by any two of its adjacent sides. Thus, the rectangle ABCD is **contained** by AB and BC and is briefly written as "the rect. AB, BC", or "AB. BC".

A four-sided figure is very often named by the two letters at opposite corners. Thus, the rectangle ABCD might be named the rect. AC or the rect. BD.

Definition

7. Any side of a parallelogram or a triangle may be called the **base.**

The **altitude** (or **height**) of a parallelogram with reference to a given side as base is the perpendicular distance between the base and the opposite side.

Thus, in fig. 1 below, CM is the altitude of the par^m AC with reference to AB as base.

The **altitude** (or **height**) of a triangle with reference to a given side as base is the perpendicular distance of the opposite vertex from the base.

For example, in fig. 2 below, CM is the altitude of the Δ ABC with reference to AB as base.

8. *Parallelograms or triangles which are between the same parallels have equal altitudes.*

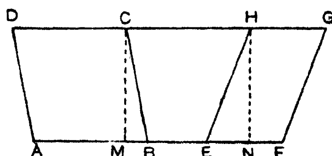


Fig. 1.

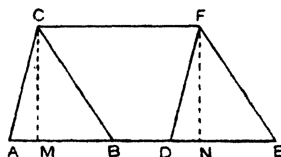


Fig. 2.

Thus, in fig. 1, the parallelograms AC, EG are between the same parallels, AF and DG. The figure CMNH is clearly a rectangle.

$\therefore$ CM = HN.

Also in fig. 2, the Δ^s ABC, DEF are between the same parallels AE and CF. The figure CMNF is evidently a rectangle.

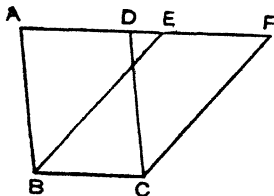
$\therefore$ CM = FN.

CHAPTER II

THEOREMS

THEOREM 27.

Parallelograms on the same base and between the same parallels (hence, of the same altitude) are equal in area.



Let the par^{ms} ABCD and EBCF be on the same base BC and between the same parallels AF, BC.

It is required to prove that the par^{ms} ABCD *and* EBCF *are equal in area.*

Proof. Since DC is ∥ to AB,

∴ the ∠FDC = the int. opp. ∠EAB.

Again, since EB is ∥ to FC,

∴ the ∠DFC = the ext. ∠AEB.

Thus, in the two Δ^s FDC, EAB, we have

the ∠FDC = the ∠EAB,

the ∠DFC = the ∠AEB,

and DC = AB; *opp. sides of a par^m.*

∴ the two Δ^s are equal in all respects. *Th. 18.*

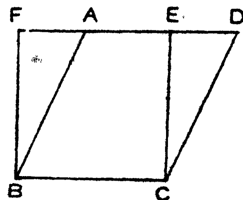
Hence, the whole quadl. ABCF *minus* the ΔFDC

= the whole quadl. ABCF *minus* the ΔEAB.

∴ the par^m ABCD = the par^m EBCF.

Q. E. D.

9. Area of a Parallelogram.



Let ABCD be a par^m on the base BC. Draw CE ⊥ to AD; then EC is the altitude of the par^m.

Draw BF ∥ to CE, meeting DA produced in F. Then FBCE is a rectangle, and it is contained by BC and CE.

Now, since FC and AC are par^{ms} on the same base and between the same parallels.

∴ **area of the par^m** ABCD
= area of the rect. BCEF
= BC × EC
= base × altitude.

COR. *Parallelograms on equal bases and of equal altitudes are equal in area.*

This follows easily, because the area of a par^m depends only on its base and altitude.

Ex. Find the area of a par^m whose base is 5 ft. and altitude 3 ft.

Area reqd. = base × altitude = 5 × 3 sq. ft. = 15 sq. ft.

EXERCISE 1

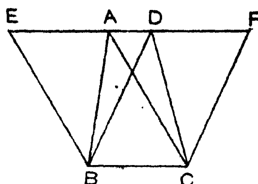
1. Divide a given parallelogram into four equal parallelograms.

2. A parallelogram and an equal rectangle stand on the same base and are on the same side of it. Show that the perimeter of the parallelogram is greater than that of the rectangle.

3. Find the area of a parallelogram two of whose adjacent sides are 4 cm. and 5 cm. and the included angle between these sides is 30°.

THEOREM 28.

Triangles on the same base and between the same parallels (hence, of the same altitude) are equal in area.



Let the Δ^s ABC, DBC be on the same base BC and between the same $\|^s$ AD, BC.

It is required to prove that the Δ^s ABC, DBC *are equal in area.*

Draw BE ∥ to CA, meeting DA produced in E; and draw CF ∥ to BD, meeting AD produced in F.

Proof. EBCA and DBCF are par^{ms}; and they are on the same base BC, and between the same $\|^s$ EF, BC.

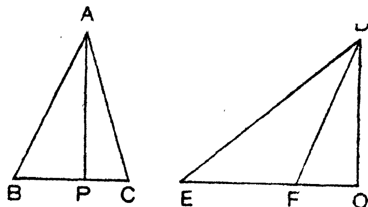
∴ the par^m EBCA = the par^m DBCF.

But, the Δ ABC = half the par^m EBCA;
and the Δ DBC = half the par^m DBCF. } *Th. 23.*

∴ the Δ ABC = the Δ DBC.

Q. E. D.

COR. *Triangles on equal bases and of the same altitude are equal in area.*



Let ABC and DEF be two Δ^s on equal bases BC and EF, and having equal altitudes AP, DQ.

Suppose the ΔDEF taken up and so placed that EF may coincide with BC, and D may fall on the same side of BC as A. Then the two Δ^s will be on the same base and of the same altitude, and hence they are equal in area.

NOTE. Hence triangles on equal bases and *between the same parallels* are equal in area. For, triangles between the same parallels are of the same altitude.

Ex. 1. Any median of a triangle divides the triangle into two equal parts.

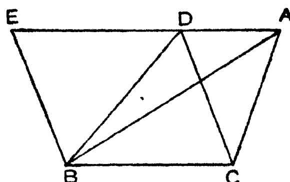
For, the triangles on the two sides of the median are on equal bases and have a common altitude.

Ex. 2. Any diagonal of a parallelogram divides the parallelogram into two equal parts.

For, the parallelogram is divided by the diagonal into two triangles which stand on equal bases and are of equal altitudes.

THEOREM 29.

If a triangle and a parallelogram be on the same base and between the same parallels, then the area of the triangle is equal to half that of the parallelogram.



Let the $\triangle$ ABC and the par^m BCDE be on the same base BC, and between the same ∥^{s} EA, BC.

It is required to prove that the $\triangle$ ABC *is half of the par^m* BCDE.

Proof. Join BD.

The diagonal BD bisects the par^m EC;

∴ the $\triangle$ DBC is half of the par^m BCDE.

But the $\triangle$ DBC = the $\triangle$ ABC; because they are on the same base BC, and between the same ∥^{s} BC, DA;

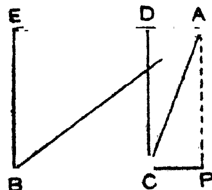
∴ the $\triangle$ ABC is half of the par^m BCDE.

Q. E. D.

COR. *The area of a triangle is equal to half that of a rectangle contained by the base and altitude of the triangle.*

Let ABC be a $\triangle$ of which BC is the base and AP, the altitude.

Draw AE ∥ to PB; also draw CD and BE each ∥ to PA, meeting AE in D and E respectively.



Then the par^m EC and the $\triangle$ ABC are on the same base and between the same ∥^{s}, ∴ the $\triangle$ ABC = $\frac{1}{2}$ of the par^m EC.

But, since CD is ∥ to PA, the par^m EC is evidently a *rectangle*; and it is contained by BC and CD, *i.e.*, by BC and PA.

∴ the △ABC = ½ of the rect. contained by BC and PA.

10. The area of a triangle.

Let BC = a units and the altitude PA = p units of length.

∴ **area of the triangle ABC**

$= \frac{1}{2}$ of the area of the rect. contained by BC, PA

$= \frac{1}{2}.BC \times PA = \frac{1}{2}\, ap$ units of area.

The result may also be expressed as follows :—

(i) **area of a triangle**

$= \frac{1}{2}$.base × altitude.

(ii) *altitude = twice the area ÷ base.*

(iii) *base = twice the area ÷ altitude.*

Ex. 1. Find the area of a triangle whose base and altitude are 7 cms. and 4 cms. respectively.

The reqd. area $= \frac{1}{2}$. base × altitude

$= \frac{1}{2} \times 7 \times 4$ sq. cm.

$= 14$ sq. cm.

Ex. 2. The area of a triangle is 54 sq. ft. If the altitude is 18 ft. find the base.

The base required

$$= \frac{2 \times \text{area}}{\text{altitude}} = \frac{2 \times 54}{18} \text{ ft.}$$

$= 6$ ft.

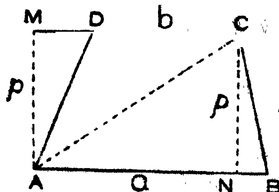
11. Area of a trapezium.

Let ABCD be a trapezium with AB par^l to CD.

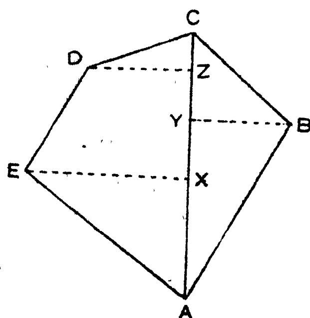
Let AB = a units of length

CD = b units of length.

Draw AM perpendicular on CD and CN perpendicular on AB.



Example. Find the area of the figure ABCDE from the following plan and table of measurements.



	Yards	
	AC = 61	
DB = 28	AZ = 50	
	AY = 40	BY = 22
EX = 40	AX = 20	

Here the fig. = $\triangle$ABC + $\triangle$DCZ + $\triangle$AEX + trap^m DEXZ.

Now, $\triangle ABC = \frac{1}{2}.\ AC.\ BY = \frac{1}{2} \times 61 \times 22 = 671$ sq. yds.

$\triangle DCZ = \frac{1}{2}.\ CZ.\ DZ = \frac{1}{2} \times 11 \times 28 = 154$ sq. yds.

$\triangle AEX = \frac{1}{2}.\ AX.\ BX = \frac{1}{2} \times 20 \times 40 = 400$ sq. yds.

trap^m $DEXZ = \frac{1}{2}.\ (DZ + EX).XZ = \frac{1}{2} \times 68 \times 30 = 1020$ sq. yds.

$\therefore$ Adding up, the total area reqd. = 2245 sq. yds.

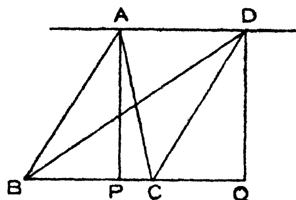
NOTE. The above table is briefly recorded as follows :

	Yards	
	To C	
	61	
To D 28	50	
	40	22 to B
To E 40	20	
	From A	

The distances are measured from A along the base line to the points from which the offsets spring and are recorded in the central column. The measures of the offsets are recorded on the right or on the left of this column according as they are the right or to the left of the base line.

THEOREM 30.

Equal triangles on the same base are of the same altitude.



Let the Δ^s ABC and DBC, standing on the same base BC, be equal in area.

Let AP and DQ be the altitudes of the Δ^s ABC and DBC.

It is required to prove that AP = DQ.

Proof. The Δ ABC = $\frac{1}{2}$ the rect. BC, AP
and the Δ DBC = $\frac{1}{2}$ the rect. BC, DQ
$\therefore$ the rect. BC, AP = the rect. BC, DQ
$\therefore$ AP = DQ. Q. E. D.

NOTE. It may be similarly proved that *equal triangles on equal bases are of the same altitude.*

COR. *Equal triangles on the same base (or on equal bases in the same straight line) and on the same side of it are between the same parallels.*

For in the above diagram, AP being equal and ∥ to DQ, AD is ∥ to PQ, *i.e.*, to BC.

EXERCISE 2

If A, a, p are the area, base and altitude of a triangle respectively, find

1. A when a=12 cm.; p=7 cm.
2. a when A=144 sq. ft.; p=24 ft.
3. p when a=12 in.; A=30 sq. in.
4. A when a=5p; p=3 ft.
5. a when A=125 sq. cm.; p=10a.

6. Find the area of a par^m if the base=15 ft. and height= 3 yds.

7. Find the area of a rhombus if its diagonals are (i) 7 cm. and 20 cm.; (ii) 8 ft., 17 ft.; (iii) 5 yds., 12 ft.

8. Find the area of a rectangle if the sides are (i) 10 cm. and 7 cm.; (ii) 23 ft., 5 ft.; (iii) 18 yds., 13 ft.

9. Find the area of a square when the side is (i) 5 in.; (ii) 7 cm.; (iii) 3 yds.

10. Find the area of a square if a diagonal is (i) 7 ft.; (ii) 12 cm.; (iii) 13 yds.

11. Prove that the four triangles into which a parallelogram is divided by its diagonals are equal in area.

12. Prove that equal triangles on opposite sides of the same base have equal altitudes. Hence prove that if two equal triangles ABC, DBC be on opposite sides of the same base BC, AD is bisected by BC or BC produced.

13. BC and AD are the parallel sides of a trapezium and E is the mid-point of CD. Prove that the ΔAEB is half of the whole figure.

14. If two triangles have two sides of the one equal to two sides of the other, and the contained angles supplementary, show that the two triangles can be so placed as to form two parts of the same triangle, and hence that they are equal in area.

15. If two sides of a triangle are given, prove that the area is greatest when the included angle is a right angle.

16. Prove that if a quadrilateral be bisected by each of its diagonals, it must be a parallelogram.

17. Prove that in any quadrilateral the straight lines joining the middle points of opposite sides bisect each other.

18. If E, F be the mid-points of the sides AC, AB of a triangle ABC, prove that the triangle AEF is one-fourth of the whole triangle ABC.

19. D, E are the mid-points of the sides AB, AC of a triangle ABC; and P is any point in BC produced. Prove that the triangle PDE is one-fourth of the triangle ABC.

20. If two parallelograms have two adjacent sides of the one respectively equal to two adjacent sides of the other, and the included angles supplementary, prove that the parallelograms are equal in area.

21. On the base of a given triangle, draw an isosceles triangle of equal area.

22. If the middle points of the sides of a quadrilateral are joined in order, the parallelogram so formed is half the quadrilateral.

23. ABCD is a par^m. If any point P be taken within it, prove that the Δ^s PAB, PCD are together equal to half the par^m.

24. AD is a median of the ΔABC. If P be any point on AD, prove that the ΔPAB = the ΔPAC.

25. Prove that the area of a rhombus is half the product of its diagonals.

26. The base of a triangle is divided into three equal parts. Prove that lines drawn from the vertex to the points of division divide the triangle into 3 equal parts.

27. Prove that the sum of the distances of any point within an equilateral triangle from its sides is the same for all positions of the point.

28. Prove that the sum of the distances of any point within a rhombus from its sides is the same for all positions of the point.

29. On the base AB construct a par^m ABEF equal in area to a given par^m ABCD such that the $\angle$ABE is 45°.

30. On the base AB construct an isosceles triangle equal to a given par^m ABCD.

31. The sides of a rt. angled triangle are 5, 12 and 13 ft. Find its height with reference to the hypotenuse as base.

32. Find the area of a trapezium if the parallel sides and the altitude are (i) 11 ft., 9 ft.; 8 ft.;
(ii) 13 ft., 19 ft; 2 yds.:
(iii) 9 cm., 21 cm.; 3 cm.

33. Prove that any str. line through the point of intersection of the diagonals of a parallelogram bisects it. Hence show how a parallelogram may be bisected by a str. line drawn
(i) through a given point;
(ii) par^l to a given line;
(iii) perpendicular to a given line.

34. Find the locus of the vertex of a triangle on a fixed base and having a constant area.

35. Find the locus of the point of intersection of the diagonals of a parallelogram on a given fixed base and having a constant area.

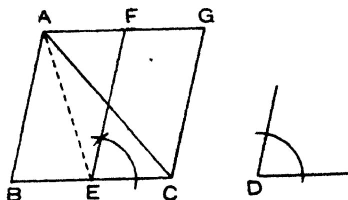
36. A square and a rhombus stand on the same base. Prove that the rhombus has a lesser area.

CHAPTER III

PROBLEMS ON AREAS

PROBLEM 17.

To construct a parallelogram equal to a given triangle and having one of its angles equal to a given angle.



Let ABC be a given △, and D the given ∠.

It is required to construct a par^m equal to the △ABC, *and having one of its* ∠^{s} *= the* ∠D.

Construction. Bisect BC at E.

At E in CE, make the ∠CEF = the ∠D; draw CG ∥ to EF; draw AG ∥ to BC, meeting EF and CG in F and G respectively.

Then the figure FC is the required par^m.

Proof. Join AE.

Since the two △^{s} ABE, AEC are on equal bases BE, EC, and of the same altitude;

∴ the △ABE = the △AEC.

∴ the △ABC is double of the △AEC.......... (α)

The quadl. FC is a par^m, because its opposite sides are ∥.

Now, the par^m FC and the △AEC are on the same base EC and between the same ∥^{s} AG, EC;

∴ the par^m FC is double of the △AEC................ (β)

Hence, from (α) and (β), the par^m FC = the △ABC;

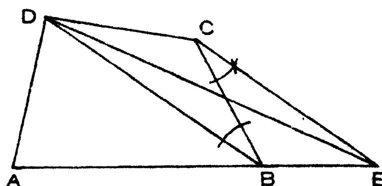
and it has also the ∠CEF = the ∠D. Q. E. F.

COR. **To construct a rectangle equal in area to a given triangle.**

If in the above problem, ∠D = 90°, the par^m constructed will be a rectangle.

PROBLEM 18.

To construct a triangle equal in area to a given quadrilateral.



Let ABCD be the given quadrilateral.

It is required to construct a triangle equal in area to ABCD.

Construction. Join DB.

From C draw CE par^l to DB meeting AB produced at E.

Join DE.

Then AED is the required triangle.

Proof. Because the Δ^s BDE, BDC are on the same base BD and between the same parallels DB, CE,

∴ the Δ BDE = the Δ BDC in area.

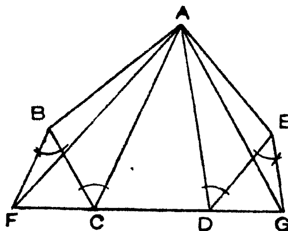
To each of these equals add the Δ ABD ;

then the Δ AED = the fig. ABCD.

Q. E. F.

PROBLEM 19.

To construct a triangle equal in area to any given rectilineal figure.



Let ABCDE be a given rectilineal figure.

It is required to construct a △ *equal in area to the rectilineal figure* ABCDE.

Construction. Join AC, AD.

Draw BF ∥ to AC, meeting DC produced in F: and draw EG ∥ to AD, meeting CD produced in G.

Join AF, AG.

Then AFG is the required △.

Proof. The △[s] ABC, AFC are on the same base AC and between the same ∥[s] AC, BF;

∴ the △ABC = the △AFC........................(α)

Also, the △[s] AED, AGD are on the same base AD and between the same ∥[s] AD, EG;

∴ the △AED = the △AGD........................(β)

Hence, from (α) and (β),

the △AFC + the △ACD + the △AGD
= the △ABC + the △ACD + the △AED;

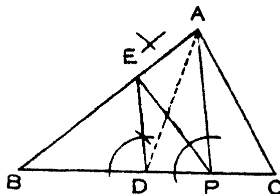
i.e, the △AFG = the rectil. figure ABCDE. Q. E. F.

NOTE 1. The given rectil. figure is = the rectil. figure ABCG which has one side less than those of the given figure. From this it is clear that a *seven* sided rectil. figure can be reduced to one of *six* sides, and this latter again to one of *five* sides; and so on. Thus, by successive repetitions of the above process, any given rectil. figure may be finally reduced to a triangle of equal area.

NOTE 2. We can construct a *rectangle* equal to any given rectil. figure. For, all that we have to do is to reduce the given figure to a triangle of equal area, and then construct a rectangle equal to this triangle. (See *Prop.* 17.)

PROBLEM 20.

To bisect a triangle by a straight line drawn through a given point in one of its sides.



Let ABC be the given triangle, and P the given point in the side BC.

It is required to bisect the △ ABC *by a str. line through* P.

Construction. Join AP.

Through D, the middle point of BC, draw DE par' to PA meeting AB at E.

Join EP.

Then PE bisects the △ ABC.

Proof. Join AD.

Because the △s EDP, EDA are on the same base ED and between the same parallels ED, AP.

∴ the △ EDP = the △ EDA in area.

To each of these equals add the △ EBD.

Then, the △ EBP = the △ ABD.

But BD = DC

∴ the △ ABD = the △ ACD

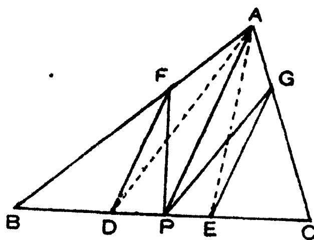
$= \frac{1}{2}$ the △ ABC.

Hence, the △ EBP = $\frac{1}{2}$ the △ ABC.

Q. E. F.

PROBLEM 21.

To trisect a triangle by straight lines drawn through a given point in one of its sides.



Let ABC be the given triangle and P the given point in the side BC.

It is required to trisect the △ABC *by straight lines drawn through* P.

Construction. Join AP.

Trisect BC at D, E and through D, E draw lines DF, EG par[l] to PA, meeting the sides BA, CA at F and G respectively.

Join PF, PG.

Then, PF and PG trisect the △ABC.

Proof. Join AD, AE.

Because the △[s] PDF, ADF stand on the same base DF and are between the same par[ls] DF, PA;

∴ the △PDF = the △ADF.

To each of these equals add the △BDF.

Then, the △BPF = the △ABD.

But, ∵ BD = DE = EC,

∴ the △ABD = the △ADE = the △AEC = $\frac{1}{3}$ △ABC. *Th. 28, Cor.*

Hence the △BPF = $\frac{1}{3}$ △ABC.

Similarly, the △CPG = $\frac{1}{3}$ △ABC.

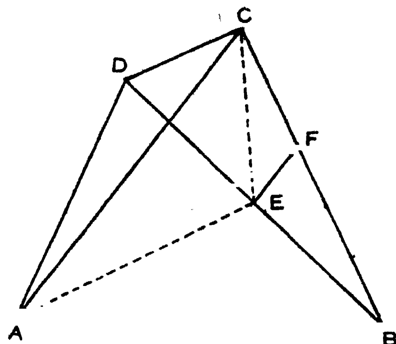
∴ the remaining figure AFPG = $\frac{1}{3}$ △ABC;

i.e., the lines PF, PG trisect the △ABC. Q. E. F.

N.B. **Students are always required to show distinctly all traces of constructions.**

PROBLEM 22.

To bisect a quadrilateral by a str. line drawn through an angular point.



Let ABCD be the given quadrilateral and A the given angular point.

It is required to bisect ABCD *by a str. line drawn through* A.

Construction. Join AC, BD.

Let the Δ ABC be $>$ Δ ADC.

Bisect BD at E and through E draw EF par^l to AC meeting BC at F.

Join AF.

Then the line AF bisects the quadrilateral ABCD.

Proof. Join AE, CE.

Because the Δ^s ACE, ACF stand on the same base AC and are between the same par^{ls} AC, EF.

$\therefore$ Δ ACF $=$ Δ ACE.

To each of these equals add the Δ ACD.

Then the fig. ADCF $=$ the fig. ADCE.

But the fig. ADCE $=$ Δ ADE $+$ Δ CDE

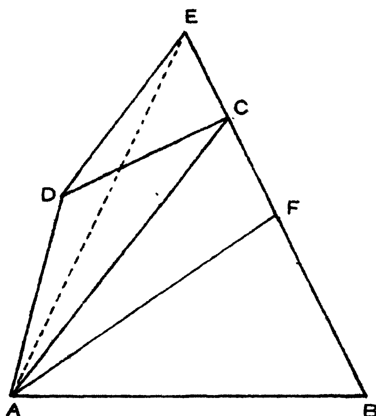
$= \frac{1}{2}\Delta$ ABD $+ \frac{1}{2}\Delta$ CBD [$\because$ DE $= \frac{1}{2}$ BD.]

$= \frac{1}{2}$. quadl. ABCD.

$\therefore$ the fig. ADCF $= \frac{1}{2}$. quadl. ABCD.

i.e., AF bisects the quadl. ABCD. Q. E. F.

PROBLEM 22. SECOND METHOD.



Construction. Join AC.

Let the Δ ABC be $>$ Δ ACD.

Draw the Δ ABE = the quadl. ABCD.

Prob. 18.

Bisect BE at F.

Join AF.

Then AF bisects the quadl. ABCD.

Proof. Since BF $= \frac{1}{2}$ BE, $\therefore$ Δ ABF $= \frac{1}{2}$ Δ ABE $= \frac{1}{2}$. quadl. ABCD,

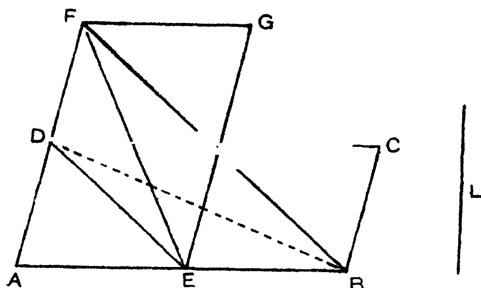
i.e., AF bisects the quadl. ABCD.

Q. E. F.

NOTE. The condition that ΔABC is $>$$\Delta$ADC is necessary; otherwise F would lie on BC produced, in which case the above construction fails.

PROBLEM 23.

To construct a parallelogram equal in area to a given parallelogram having one side of given length.



Let ABCD be the given par^m and L the given side.

It is required to construct on a side = L, *a par*^m *equal in area to* ABCD.

Construction. From AB cut off AE = L. Join ED.

Through B draw BF par^l to ED meeting AD produced at F.

Through F draw FG par^l to AE and through E draw EG par^l to AF, to meet FG at G.

Then AEGF is a required par^m.

Proof. Join EF, BD.

Now, the par^m AEGF has the side AE = L.

Also, ∵ the Δ^s DEF, BED stand on the same base DE and are between the same par^{ls} ED, BF.

∴ the Δ DEF = the Δ BED.

To each of these equals add the Δ AED,

∴ the Δ AEF = the Δ ABD.

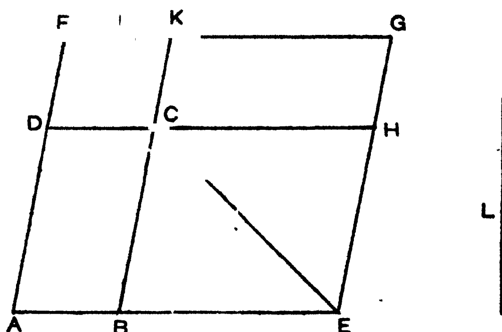
But, par^m AEGF = 2 Δ AEF and par^m ABCD = 2 Δ ABD, *Th.* 23.

∴ par^m AEGF = par^m ABCD. Q. E. F.

N. B. Notice that any other par^m which has the same base AE and is between the same par^{ls} AE, FG is also a required par^m.

NOTE. If the required par^m must have also an angle equal to a given angle, draw the line AH such that ∠EAH = the given angle. Let AH meet FG at H. Complete the par^m AEIH, which is the reqd. par^m.

PROBLEM 23. SECOND METHOD.



Construction. Produce AB to E so that BE = L. Join EC and produce it to meet AD produced at F. Complete the par^{m} AEGF.

Produce BC to meet FG at K and DC to meet EG at H.

Then KCHG is the required par^{m}.

Proof. Because the line ECF bisects the par^{ms} BH, DK, AG.

∴ the ΔHCE = the ΔBCE ;
the ΔKCF = the ΔDCF ;
the ΔGEF = the ΔAEF.

Hence, ΔGEF − ΔHCE − ΔKCF = ΔAEF − ΔBCE − ΔDCF.

∴ par^{m} KCHG = par^{m} ABCD.

Also, the side CH = BE = L. Q. E. F.

15. In the above figure the par^{ms} BH, DK are called **parallelograms about the diagonal** EF and the par^{ms} AC, CG are called their **complements**.

N. B. Notice that *complements of the parallelograms which are about the diagonal of a par^{m} are equal.*

Thus, in the above figure, par^{ms} AC, CG have been proved to be equal.

EXERCISE 8

1. On the side of a parallelogram draw a parallelogram of equal area having an angle equal to (i) 45° ; (ii) 60° ; (iii) 90° ; (iv) 120° ; (v) 135°.

2. On the base of a triangle draw a triangle of equal area having an angle at the base equal to a given angle.

3. Construct a triangle whose sides are 3·5 cm., 4 cm. and 3 cm., and transform it (i) into an equivalent right-angled triangle and (ii) into an equivalent isosceles triangle having the base 4 cm. in each case.

4. On one side of a triangle construct a parallelogram of equal area having one of its angles equal to a given angle.

5. On one side of a triangle as base construct an isosceles triangle equal to half the given triangle.

Hence show how to construct a rhombus equal in area to a triangle having one side of the triangle as a diagonal.

6. P is a given point in the side AB of a triangle ABC. Find a point D in BC produced so that the triangle PBD may be equal to the triangle ABC.

7. ABC is a triangle and P is a point in BA produced. Find a point D in BC so that the triangle PBD may be equal to the triangle ABC.

8. On a *given base* construct a triangle equal in area to a given triangle.

[Let ABC be the given Δ and X, the given base. Cut off from BC, (or BC produced if necessary), BD equal to X. Join AD. Through C draw CP || to DA meeting BA or BA produced in P. Then ΔPBD is one of the required triangles.]

9. The sides of a triangle are 10 cm., 12 cm., 14 cm. Construct a triangle of equal area, one of whose sides is 9 cm.

10. Draw a triangle whose sides are 5″, 6″, ·7·5″. Draw an equivalent triangle, one of whose sides is 10″. [1 inch is written as 1″.]

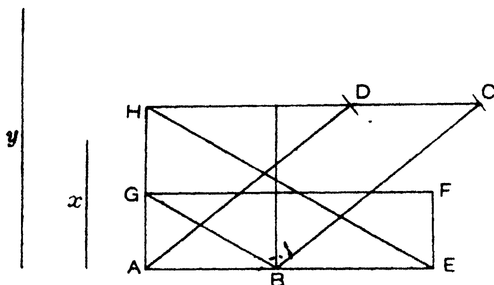
11. On a given base draw an isosceles triangle equal in area to a given triangle.

12. On a given base describe a triangle having a given angle at the base and equal in area to a given triangle.

13. On a given base AB construct a rectangle equal to a given square.

14. On a given base AB construct a rectangle equal to a given rectangle.

15. Draw a parallelogram ABCD with its sides AB, AD equal to given straight lines and its area equal to that of a given rectangle AEFG.



Let x, y be the given str. lines and suppose AB$=x$ and AD$=y$.

Along AE or AE produced set off AB$=x$. Join BG. Draw EH parallel to BG to cut AG or AG produced in H. Draw HDC parallel to AE. With centre as A and radius AD ($=y$) draw an arc cutting HDC at D. Complete the par^m ABCD.

Evidently, the par^m AC=the rect. HB=the rect. AF, by Ex. 14.

16. On a given base AB construct a par^m equal in area to a given par^m and having one of its angles equal to a given angle.

17. On a given str. line construct a par^m equivalent to a given triangle and having one of its angles equal to a given angle.

[On double the given base, construct a Δ equivalent to the given triangle (See Ex. 8). Then apply *Prob.* 17.]

18. On a given base construct a rectangle equivalent to a given triangle.

19. Draw a par^m with sides 5″, 3″ and the contained angle 30°. On the same base construct a rhombus equal to its area, noting the case of failure, if any.

20. Describe a par^{m} (i) equal in area to a given quadrilateral ;
(ii) equivalent to a given rectilineal figure,
and having an angle equal to a given angle.

21. Draw a rectangle equal to a given (i) quadrilateral ;
(ii) pentagon.

22. Draw a rhombus equal to a given (i) triangle; (ii) quadrilateral and (iii) pentagon.

23. Divide a triangle into five equal parts by lines drawn through one of its angular points.

24. Divide a triangle into two parts in the ratio 1 : 4 by a str. line drawn through a point in any one of its side.

25. Divide a triangle into two parts in the ratio 2 : 3 by a line drawn through a vertex.

26. Divide a triangle into two parts in the ratio 3 : 4 by a line drawn through any point in a side.

27. Divide a triangle into any number of equal parts by drawing lines from a vertex.

Hence, divide a given triangle into two parts in the ratio $m : n$ by a line (i) drawn through a vertex ;
(ii) drawn through any point in its side.

28. From a quadrilateral cut off a third part by drawing a line through one of its vertices.

[In Prob. 22, 2nd *Method*, cut off BF = $\frac{1}{3}$BE, then, ΔABF is clearly a third part.]

29. Draw a line through a vertex of a quadrilateral dividing it into two parts in the ratio 2 : 3.

30. *Bisect a quadrilateral by a str. line drawn through a given point in one of the sides.*

[With the given pt. as vertex, draw a triangle=quadl. (See Prob. 18) and join the pt. with the mid-pt. of its base.]

31. Trisect a parallelogram by lines drawn through an angular point.

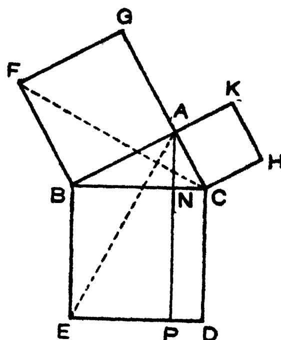
32. Divide a par^{m} into two parts in the ratio 2 : 3 by a line drawn through an angular point.

CHAPTER IV

PYTHAGORAS' THEOREM

THEOREM 31.

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the sides containing the right angle.



Let ABC be a right-angled Δ, having the right angle at A. Let BCDE, ABFG and ACHK be the squares described on BC, AB and AC respectively.

It is required to prove that the sq. BD = *the sum of the squares* AF *and* AH.

Draw AP ∥ to BE or CD, meeting DE in P.
Join, AE, CF.

Proof. Since each of the adj. ∠ˢ BAC, BAG is a rt. ∠,
∴ CA and AG are in one str. line.

Now, the ∠CBE = the ∠ABF; [*each being a rt.* ∠]
∴ adding the ∠ABC to each of them, we have the whole ∠ABE = the whole ∠FBC.

Then in the two △ˢ ABE, FBC, we have

AB = FB,

BE = BC,

and the ∠ABE = the ∠FBC;

∴ the two △ˢ are equal in all respects.

Now, the △ABE and the rect. BP are on the same base BE and between the same ∥ˢ AP, BE;

∴ the rect. BP = twice the △ABE.

Also, since the △FBC and the sq. AF are on the same base FB and between the same ∥ˢ GC, FB,

∴ the sq. AF = twice the △FBC.

Hence, the rect. BP = the sq. AF.

Similarly, by joining AD and BH, it may be shown that the rect. CP = the sq. AH.

∴ the rectangle BP and CP are together
= the sum of the squares AF and AH,

i.e., the sq. BD = the sum of the sqs. AF and AH.

Q. E. D.

NOTE. This theorem was discovered by **Pythagoras**.

16. The above result may be briefly written as follows:

$$BC^2 = CA^2 + AB^2.$$

i.e., if a and b denote the lengths of the sides containing the right angle and c = the length of the hypotenuse

$$c^2 = a^2 + b^2,$$

Hence, $a^2 = c^2 - b^2$,

and $b^2 = c^2 - a^2$.

Therefore, *when any two sides of a right-angled triangle are known the third side can be found out.*

Ex. 1. If the sides of a right angled triangle containing the rt. angle are 3 cm. and 4 cm., find its hypotenuse.

Let the hypotenuse $= c$ cm.

$$\therefore \quad c^2 = 3^2 + 4^2 = 9 + 16 = 25$$

$$i.e., \quad c = \sqrt{25} = 5.$$

Hence, the hyp. $= 5$ cm.

Ex. 2. A ladder 13 ft. long is placed with one of its ends on a horizontal plane so as to reach the top of a vertical wall 12 ft. high. How far is the foot of the ladder from wall ?

Let the distance reqd. $= x$ ft.

Then, clearly the ladder forms the hypotenuse of a right-angled triangle whose sides containing the rt. $\angle$ are 12 ft. and x ft.

$$\therefore \quad 13^2 = 12^2 + x^2$$
$$\text{or,} \quad 169 = 144 + x^2$$
$$\text{or,} \quad x^2 = 169 - 144 = 25;$$
$$\therefore \quad x = \sqrt{25} = 5.$$

17. In the course of the proof of Th. 31, the following important results have been established :

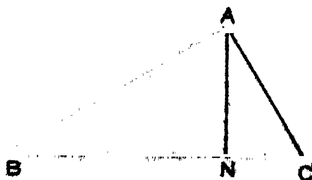
If BC and AP intersect at N, then it has been proved that the sq. AF $=$ rect. BP

i.e., the sq. on AB $=$ the rect. contained by BN, BC ; [$\because$ BE $=$ BC]

Also, the sq. AH $=$ the rect. CP,

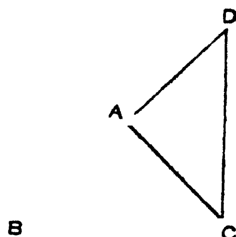
i.e., sq. on AC $=$ the rect. contained by CN, BC. [$\because$ CD $=$ BC]

These results may also be expressed thus: If ABC be a triangle right-angled at A and AN is drawn perpendicular on the hypotenuse BC, then $AB^2 = BN.BC$ and $AC^2 = CN.BC$.



THEOREM 32.

If the square described on one side of a triangle be equal to the sum of the squares described on the other two sides, then the angle contained by those two sides is a right angle.



Let ABC be a triangle such that $BC^2 = CA^2 + AB^2$.

It is required to prove that the $\angle BAC$ *is a right angle.*

Draw AD ⊥ to AC, and make AD = AB.

Join DC.

Proof. Since the $\angle CAD$ is a rt. $\angle$,

$$\therefore \quad DC^2 = CA^2 + AD^2$$
$$= CA^2 + AB^2.$$

Hence the sq. on DC = the sq. on BC;

$$\therefore \quad DC = BC.$$

Now, in the two Δ^s ABC, ADC, we have

AB = AD, *Cons.*

AC common,

and BC = DC,

∴ the two Δ^s are congruent.

Hence the $\angle BAC$ = the $\angle DAC$ = a rt. $\angle$. Q. E. D.

Ex. 1. We have $3^2 + 4^2 = 5^2$. Hence if the sides of a triangle are 3, 4 and 5 units of length respectively the triangle is right-angled.

Ex. 2. Again, since $5^2+12^2=13^2$, $\therefore$ the triangle whose sides are 5, 12 and 13 units of length respectively, is right-angled.

Ex. 3. We have $(m^2-n^2)^2+(2mn)^2=(m^2+n^2)^2$.

$\therefore$ if the sides of a triangle are m^2-n^2, $2mn$ and m^2+n^2 units of length respectively, the triangle is right-angled.

Hence for each pair of numbers that may be substituted for m and n, we shall get a set of three numbers representing three str. lines which can form a right-angled triangle. Thus if $m=2$ and $n=1$, we have the sides equal to 3, 4 and 5 units respectively; if $m=4$ and $n=3$, we have the sides equal to 7, 24 and 25 units respectively; if $m=4$ and $n=1$, we have the sides equal to 15, 8 and 17 units respectively; and so on.

EXERCISE 4

1. Find the hypotenuse of a right-angled triangle if the sides containing the rt. $\angle$ are
(i) 3, 4; (ii) 5, 12; (iii) 8, 15;
(iv) 24, 10; (v) 7, 24; (vi) 11, 17.

2. If the hypotenuse and a side of a rt. angled triangle are (i) 25, 7; (ii) 17, 8; (iii) 26, 10; (iv) 25, 24; (v) 13, 10; find the third side in each case.

3. Prove that a triangle is right angled if its sides are
(i) 6, 8, 10; (ii) 10, 24, 26; (iii) 14, 48, 50;
(v) 16, 30, 34; (v) 15, 20, 25; (vi) 28, 45, 53.

4. If a man starting from a place A travels 12 miles due North and then 5 miles due East, how far is he from A at the end of his journey?

5. A ladder 37 feet long rests with one end on the ground and the other on the top of a wall. Calculate the height of the wall if the distance of the foot of the ladder from the wall be 12 feet.

6. Two posts 25 ft. and 32 ft. high are 24 ft. apart. Find the distance between their tops.

7. A rope 68 ft. long is tied to the top of a vertical post. When stretched tight, it touches the ground at a distance of 32 ft. from the foot of the post. Find the height of the post.

8. A rod 13 feet long is held vertically in a tank of water, one end of the rod reaching the bottom of the tank, and the other being *above* the surface of the water. Another rod of equal length is held in a slant position, one end being in contact with the lower end of the first rod and the other being *on the surface* of the water. If the upper end of the second rod be 5 feet distant from the point where the first cuts the surface of water, calculate the length of that portion of the first rod which is above the water.

9. Find the diagonal of a square if its side is (i) 4 ft.; (ii) 32 cm.; (iii) 15 yds.; (iv) a ft.

10. Find the diagonal of a rectangle whose adjacent sides are (i) 5 ft., 12 ft.; (ii) 8 cm., 15 cm.; (iii) a cm., b cm.

11. If the diagonal of a rectangle is 26 ft. and its length 24 ft., calculate its area.

12. Find the side of a rhombus if its diagonals are 21 cm., 72 cm.

13. If the base and altitude of an isosceles triangle are 4·8 cm. and 4·5 cm., find the sides.

14. If the base and one of the equal sides of an isso. Δ are 4·2 ft. and 7·5 ft. respectively, find its altitude and area.

15. Find the altitude of an equilateral triangle if its side is 15 ft. Hence find its area.

16. AD is the altitude of an equilateral triangle ABC. If AD and BD respectively contain p and m units of length, prove that $p^2 = 3m^2$.

17. A square is half the square on its diagonal.

Hence construct a square (i) double a given square, and (ii) half a given square.

18. Prove that the sum of the squares on the sides of a rhombus is equal to the sum of the squares on its diagonals.

19. ABCD is a quadrilateral in which the diagonals are at right angles. Prove that

$$AB^2 + CD^2 = BC^2 + AD^2.$$

20. ABC is a triangle right-angled at A, and D is a point on AC. Prove that

$$BC^2 + AD^2 = BD^2 + AC^2.$$

21. Construct a rectangle such that the square on its diagonal is equal to five times a given square.

22. ABC is a triangle right-angled at A. AN is drawn perp. on BC, meeting BC at N. Prove that

(i) $AB^2 - AC^2 = BN^2 - CN^2$;
(ii) $AB^2 = BN.BC$;
(iii) $AC^2 = CN.BC$.

23. Construct a square equal in area to the sum of two given squares.

[Draw a rt. angled Δ with the sides of the given square as the sides containing the rt. $\angle$. The sq. on the hypotenuse is the sq. reqd.]

24. Construct a square equal in area to the difference of two given squares.

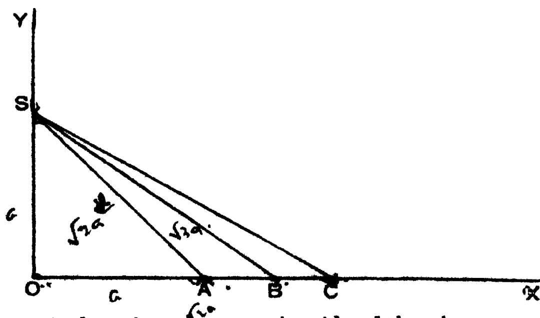
25. Divide a given straight line into two parts so that the sum of the squares on the parts may be equal to a given square.

26. Divide a given str. line into two parts such that the difference of the squares on the parts may be equal to a given square.

Some Applications of the Theorem of Pythagoras

1. *To draw squares whose areas shall be respectively twice, three times, four times etc. that of a given square.*

Hence to deduce graphically the values of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, *etc.*



Let unit length represent the side of the given square.

Take two lines OX, OY at right angles and cut off OA, OS each $=1$ unit. Join AS.

Then, $AS^2 = OA^2 + OS^2 = 1 + 1 = 2$,

i.e., sq. on AS is twice the given squares.

Also, $AS = \sqrt{2}$.

Next, cut off from OX, $OB = SA$ and join SB.

Then, $SB^2 = OB^2 + OS^2 = AS^2 + OS^2 = 2 + 1 = 3$,

i.e., the sq. on SB is three times the given square.

Also, $SB = \sqrt{3}$.

Again, cut off from OX, $OC = SB$ and join SC.

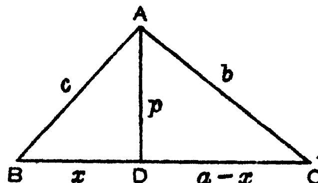
Then $SC^2 = OC^2 + OS^2 = SB^2 + OS^2 = 3 + 1 = 4,$

i.e., sq. on SC is four times the given square.

Also. $SC = \sqrt{4}$.

SA, SB, SC may be obtained by measurement and consequently $\sqrt{2}$, $\sqrt{3}$, etc. may be found. Proceeding in this manner, the values of $\sqrt{5}$, $\sqrt{6}$,......may also be determined.

2. *If the sides* BC, CA, AB *of a triangle be respectively equal to* a, b *and* c *units of length, and if* $2s = a + b + c$, *prove that the area of the triangle* $= \sqrt{s(s-a)(s-b)(s-c)}$ *units of area.*



Let the altitude AD of the triangle contain p units of length.

$\therefore$ The area of the $\Delta = \frac{ap}{2}$ units of area.

Now, let BD be $= x$ units of length; then $CD = a - x$ units of length.

From the Δ ADB, we have $p^2 = c^2 - x^2$;

and from the Δ ADC, $p^2 = b^2 - (a-x)^2$.

$$\therefore \quad c^2 - x^2 = b^2 - (a-x)^2,$$

$$\text{whence} \quad x = \frac{a^2 - b^2 + c^2}{2a}.$$

Hence, $p^2 = c^2 - x^2$

$$= c^2 - \left(\frac{a^2 - b^2 + c^2}{2a}\right)^2$$

$$= \left\{c + \frac{a^2 - b^2 + c^2}{2a}\right\}\left\{c - \frac{a^2 - b^2 + c^2}{2a}\right\}$$

$$= \frac{(c+a)^2 - b^2}{2a} \cdot \frac{b^2 - (c-a)^2}{2a}$$

$$= \frac{(c+a+b)(c+a-b)(b+c-a)(b-c+a)}{4a^2}.$$

Now, since $2s = a + b + c$,

$$\left.\begin{aligned} \therefore \quad c+a-b &= (a+b+c)-2b = 2(s-b), \\ b+c-a &= (a+b+c)-2a = 2(s-a), \\ a+b-c &= (a+b+c)-2c = 2(s-c), \end{aligned}\right\}$$

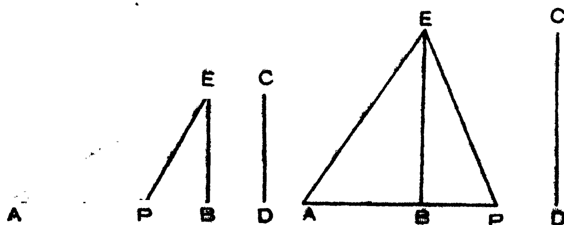
$$\therefore \quad p^2 = \frac{4s(s-a)(s-b)(s-c)}{a^2},$$

$$\therefore \quad p = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}.$$

Hence, the area of the Δ ABC

$$= \frac{ap}{2} = \sqrt{s(s-a)(s-b)(s-c)} \text{ units of area.}$$

3. AB *is a given straight line. Find a point* P *in* AB, *or* AB *produced, such that the difference of the squares on* AP, BP *may be equal to the square on a given line* CD.



Construction. Draw BE ⊥ to AB, making BE = CD.

Join AE ; and make the ∠AEP = ∠EAB.

Let EP meet AB, or AB produced, in P.

The point P thus found is the required point.

Proof. Since the ∠AEP = the ∠EAP,

$\therefore$ AP = EP.

Hence, $AP^2 - BP^2 = EP^2 - BP^2$

$= EB^2$ [$\because$ the ∠EBP is a right ∠]

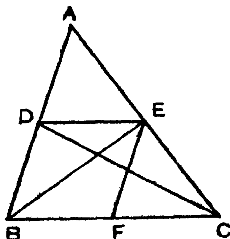
$= CD^2$.

Q. E. F.

CHAPTER V

MISCELLANEOUS PROPOSITIONS (II)

1. *The straight line joining the middle points of any two sides of a triangle is parallel to the third side and equal to half of it.*



Let D, E be the mid-pts. of the sides AB and AC of a Δ ABC.

It is required to prove that DE *is* ∥ *to* BC *and is* $= \frac{1}{2}$BC.

Proof. (1) Join BE, CD.

The Δ^s BDC, ADC are on equal bases and of the same altitude;

$\therefore$ the ΔBDC = the ΔADC;

and $\therefore$ the ΔBDC $= \frac{1}{2}$ the ΔABC

Similarly, the ΔBEC $= \frac{1}{2}$ the ΔABC.

Hence the ΔBDC = the ΔBEC.

$\therefore$ DE is ∥ to BC.

(2) *Let* F *be the mid-pt. of* BC; *join* EF.

Then since E and F are the mid-pts. of AC and BC,

$\therefore$ EF is ∥ to AB.

Hence DF is a par^m,

$\therefore$ DE = BF $= \frac{1}{2}$ BC. Q. E. D.

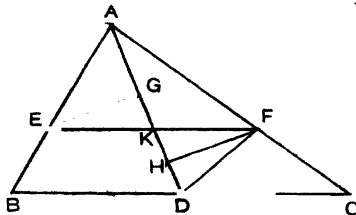
COR. *Conversely, if through the middle point of a side of a triangle a straight line be drawn parallel to another, it will pass through the middle point of the third side.*

In the above diagram, if DE be ∥ to BC, then the △BEC = the △BDC.

∴ the △BEC = ½ the △ABC.

Hence the pt. E is the mid-pt., of AC (for, if any other pt. E′ were the mid-pt.,) the △BE′C would be half of the △ABC, which is impossible.

2. *If* D *is the mid-pt. of the side* BC *of a triangle* ABC, *prove that* AD *bisects any straight line* EF, *drawn parallel to* BC *and meeting the sides* AB, AC *at* E *and* F *respectively.*



Proof. Join DE, DF.

The △ˢ BED, DFC are on equal bases BD, DC and between the same ∥ˢ EF, BC;

∴ the △BED = the △DFC.

Also the △ABD = the △ADC.

∴ the △AED = the △AFD. *Ax 3.*

Let EG and FH be the altitudes of the △ˢ AED, AFD; and let K be the pt. where EF cuts AD.

The △AED = ½ the rect. AD.EG

and the △AFD = ½ the rect. AD.FH.

∴ EG = FH.

Now, in the △ˢ EKG, FKH,

EG = FH,

the ∠EKG = the vertically opp. ∠FKH,

the ∠EGK = the ∠ FHK, *each being a rt. angle;*

∴ the two △ˢ are congruent.

Hence, EK = FK. Q. E. D.

MISCELLANEOUS EXERCISES

1. A rectangle and a par^m of equal area stand on the same base. Prove that the perimeter of the rectangle is less than that of the parallelogram.

2. The four triangles into which the diagonals divide a parallelogram are equal in area.

3. If two triangles of equal area stand on opposite sides of a common base, the line joining the vertices is bisected by the base.

4. The sum of the perpendiculars from any point within any equilateral figure to its sides is the same wherever the point is taken.

5. Of all triangles of equal area and on the same base that which is isosceles has the least perimeter.

6. ABCD is a par^m and P is any point within the angle which AD makes with BA produced. Prove that the $\triangle PAC = \triangle PAB + \triangle PAD$.

[Draw DE, BF par^l to AP to meet AC at E, F. Then $\triangle^s$ AEP, BFC being congruent, AE=CF. $\therefore$ $\triangle PCF = \triangle AEP = \triangle PAD$. Also $\triangle PAF = \triangle PAB$. Hence.]

7. If in Ex. 6, P is within the $\angle BAD$, prove that

$$\triangle PAC = \triangle PAB - \triangle PAD.$$

8. Prove that a trapezium is bisected by the line joining the middle points of the parallel sides.

9. Two sides of a triangle being given prove that the area included is greatest when the angle between the sides is a rt. $\angle$.

10. $\triangle ABC$ is drawn on a given line BC as base such that its area is equal to a given area Prove that the locus of A is a pair of lines each par^l to BC.

11. ABCD is a par^m drawn on a given line AC as diagonal such that its area is equal to a given area. Find the locus of B and D.

12. ABCP is a quadrilateral such that its area is always equal to ABCD. Find the locus of P.

13. Construct a triangle, given its area and two sides.

Is the construction always possible ?

14. Construct a triangle having given area, a side and an angle adjacent to the side.

15. Construct a rectangle equal in area to a given triangle.

16. On a given side construct a rectangle equal in area to a given rectangle.

17. Draw a parallelogram equal in area to a given parallelogram, having given also (i) a side and an angle ; (ii) two sides.

18. Show that a par^m is bisected by any line through the pt. of intersection of its diagonals.

Hence bisect a par^m by a line perpendicular to a given line.

19. Construct a triangle equal in area to a given triangle and having its altitude of given length.

20. Trisect a given triangle by lines drawn through any vertex.

21. Find the point O within the triangle ABC such that ΔAOB $= \Delta$BOC$= \Delta$COA.

22. ABCD is a quadrilateral where ΔABC is twice the ΔADC.

If AC and BD cut at O, prove that DO$=\frac{1}{3}$.BD.

23. Divide a square into three equal parts by straight lines drawn through a vertex.

24. Divide a quadrilateral into three equal parts by str. lines drawn through a vertex.

25. Divide a pentagon into three equal parts by lines drawn through a vertex.

26. Divide a pentagon into five equal parts by lines drawn through a vertex.

27. Bisect a trapezium (i) by a line drawn through one of its vertices ; (ii) by a line drawn through a point in one of the sides.

28. E, F are mid-points of the sides BC, CD of the sq. ABCD. Show that ΔAEF$=\frac{3}{5}$. trap^m ABCF.

29. Construct a parallelogram which is equal in area to a given triangle and which has a given side and a given angle.

30. If the sides containing the right angle of a right-angled triangle are a, b, find

(i) the hypotenuse ;
(ii) the area ;
(iii) the perpendicular from the rt. angle to the hypotenuse ;
and (iv) the median from the right angle to the hypotenuse.

31. If a, b are the measures of the sides of a right-angled triangle containing the rt. angle and p that of the perpendicular from the right angle to the hypotenuse, prove that

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

32. If P is any point within the rect. ABCD, PA^2+PC^2=PB^2 +PD^2.

33. A and B are two fixed points and CD any str. line perpendicular to AB meeting AB or AB produced in D. Prove that if P is any point in CD,

$$PA^2-PB^2=DA^2-DB^2.$$

34. ABC is a triangle acute angled at C. Prove that

$$AB^2 < AC^2 + BC^2.$$

Hence, prove that if in a triangle ABC,

$$AB^2 > AC^2 + BC^2,$$

then the $\angle ACB$ is obtuse.

35. The $\triangle ABC$ is rt. angled at C. If the $\angle A = 60°$, prove that (i) $AB = 2AC$; (ii) $3AB^2 = 4BC^2$; and (iii) $BC^2 = 3AC^2$.

36. If an equilateral triangle and a square stand on the same base, prove that the area of the $\triangle = \frac{\sqrt{3}}{4}$.the area of the square.

Hence deduce that the equilateral triangle described on the hypotenuse of a right-angled triangle is equal to the sum of the equilateral triangles drawn on the sides containing the right angle.

37. Divide a str. line into two parts such that the square on one part may be double the square on the other.

[If AB be the given line, construct the $\triangle ABC$ such that $\angle B = 45°$ and, $\angle A = \frac{1}{2}$ of $45°$. Draw CD making $\angle ACD = \angle A$ and meeting AB at D. Prove that $AD = CD = CB$ and $BD^2 = CD^2 + CB^2 = 2AD^2$.]

38. Divide a str. line into two parts such that the square on one part may be three times the square on the other.

[If AB be the given line, construct the $\triangle ABC$ such that $\angle B = 60°$ and $\angle A = 45°$. Draw the line CD making $\angle ACD = \angle A$, to meet AB at D. The $\triangle BDC$ is rt. angled at D and its $\angle B = 60°$. Clearly, $AD = CD$ and $AD^2 = CD^2 = 3BD^2$. See Ex. 34. (iii)]

39. Calculate the area of a triangle whose sides are (i) 10, 17, 21 cms. ; (ii) 5, 12, 13 ft (iii) 10, 11, 18 cms. ; (iv) 18, 24, 33 ft., respectively.

40. Calculate the area of a triangle whose sides are 7, 24, 25 ft. respectively.

Hence find the length of the perpendicular on the greatest side from the opposite angular point.

41. The parallel sides of a trapezium are 20 and 34 ft. and the other two sides are 13 and 15 ft. Find the area of the trapezium.

[First, find the distance between the par sides.]

42. Describe a rhombus equal to a given rectilineal figure.

43. Construct a rectangle equal to the sum or difference of two given triangles.

BOOK III

THE CIRCLE

CHAPTER I

FUNDAMENTAL IDEAS AND DEFINITIONS

1. A **circle** is a plane figure bounded by one line which is such that all straight lines drawn to it from a certain fixed point within the figure are equal to one another.

The *fixed point* is called its **centre** and the *bounding line* is called its **circumference.**

The term circle denotes the *space* bounded by the circumference; but the word is also often used to mean the *circumference* itself, if no confusion is likely to arise.

NOTE. Evidently, the circle is a **closed** figure.

2. Any straight line drawn from the centre of a circle to the circumference is called a **radius** of the circle.

Hence, (*i*) *all radii of the same circle are equal to one another*, and

(*ii*) *if any straight line be drawn from the centre of a circle equal to the radius, the extremity of the line will be on the circumference.*

3. Any straight line drawn through the centre of a circle and terminated both ways by the circumference is called a **diameter** of the circle.

Hence, a *diameter* $=2\times$ *radius*.

4. A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference which it cuts off.

Hence, every diameter divides a circle into two semi-circles.

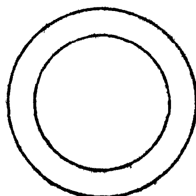
5. Two or more circles are said to be **concentric** when they have the same centre.

6. From the above definitions, the following properties of circles are evident :—

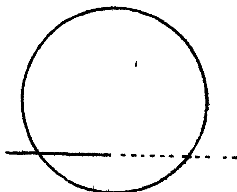
(i) *If two circles have equal radii they are congruent.*

(ii) A point is without or within a circle according as its distance from the centre is greater or less than the radius.

(iii) Concentric circles, whose radii are unequal, cannot cut one another.

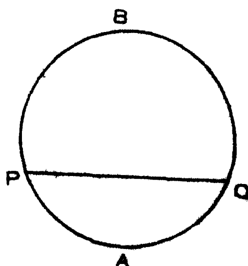


(iv) A circle is a *closed* figure ; hence, if a straight line crosses the circumference, it will also cross it at a second point when produced



7. Any part of the circumference of a circle is called an **arc** of the circle.

Thus, PAQ, PBQ are arcs



8. The finite straight line joining any two points on the circumference of a circle is called a **chord** of the circle.

Thus, in the fig. of § 7, PQ is a chord.

NOTE 1. A diameter is also a chord.

NOTE 2. Every chord which is not a diameter, divides the circumference into *unequal* arcs, of which the greater one is called the **major arc** and the lesser, the **minor arc**.

These two arcs are called **conjugate** to each other.

Thus, in the figure of § 7, PAQ is the minor arc and PBQ is the major arc.

Also the arcs PAQ, PBQ are conjugate to each other.

SYMMETRY

9. A figure is said to be **symmetrical about a line** if the part of the figure on one side of the line coincides completely with the part on the other side when the figure is folded about that line.

The line is called an **axis of symmetry** of the figure.

Ex. 1. It is easy to see that an isosceles triangle is symmetrical about the bisector of its vertical angle.

Ex. 2. A square is symmetrical about a diagonal or about each of the lines joining the middle points of its opposite sides.

Ex. 3. A rhombus is symmetrical about each of its diagonals.

If a figure is symmetrical about a line, then either half of the figure is called an **image** or **reflection** of the other half.

10. A figure is said to be symmetrical about a point O if corresponding to every point P of the figure, there is another point P′ such that the str. line PP′ is bisected at O.

Ex. 1. A circle is symmetrical about the centre.

Ex. 2. A parallelogram is symmetrical about the point of intersection of its diagonals.

11. Two points P, Q are said to be **symmetrically opposite** with respect to a line AB, if the str. line PQ is bisected at right angles by AB.

[Either of the points P, Q is called the **image** of the other in the axis of symmetry, AB.]

NOTE. Any pt. in the axis of symmetry is equidistant from the two symmetrically opposite points.

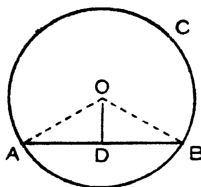
CHAPTER II

CHORDS

THEOREM 33.

If a straight line drawn from the centre of a circle bisects a chord which is not a diameter it is at right angles to the chord.

Conversely, the perpendicular to a chord from the centre bisects the chord.



Let O be the centre of the ⊙ABC and AB, a chord not passing through the centre.

Let OD bisect AB.

It is required to prove that OD *is perp. to* AB.

Join OA, OB.

Proof. In the Δ^s OAD, OBD,

∵ { OA = OB, (*radii of the* ⊙)
AD = DB, *Hyp.*
and OD is common,

∴ the two Δ^s are congruent.

Hence, the ∠ODA = the ∠ODB;

and these being adjacent angles, each = a rt. ∠.

∴ OD is perp. to AB.

Q. E. D.

(ii) *Conversely*, let OD be $\perp$ to AB.

It is required to prove that AD = DB.

Join OA, OB.

Proof. In the *right-angled* Δ^s ODA, ODB,

$\because$ { the hypotenuse OA = the hypotenuse OB ; *(being radii of the* $\odot$*)*
and OD is common

$\therefore$ the two Δ^s are congruent.

Hence AD = DB.

Q. E. D.

COR. 1. *A straight line cannot cut a circle in more than two points.*

For, if the chord AB were to meet the circle again at some other point E, then D would be the middle point of AB as well as of AE, which is impossible.

COR. 2. *The perpendicular bisector of a chord of a circle passes through its centre.*

Let PQ be a chord of a circle of which C is the centre, and let AB be the perp. bisector of PQ.

It is required to prove that AB *passes through* C.

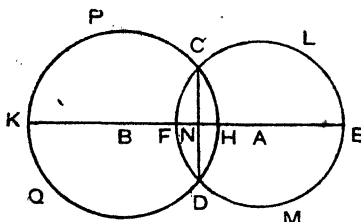
Proof. Since AB is the perp. bisector of PQ, it is the locus of points equidistant from P and Q. (*i.e.* every pt., equidistant from P and Q, lies on AB). Also the centre, C, of the circle is equidistant from P and Q. $\therefore$ C must lie on AB.

i.e. AB passes through C.

COR. 3. *The middle points of any system of parallel chords of a circle lie on a straight line passing through the centre.*

For, if a str. line be drawn from the centre $\perp$ to one of the chords, it will be $\perp$ to all the chords, and will $\therefore$ intersect all of them at their middle points.

Two important Theorems : *(1) If two circles cut each other the line joining their centres bisects the common chord at right angles.*



Let the two ⊙ˢ whose centres are A and B cut each other at the points C and D, so that CD is their common chord.

It is required to prove that AB *bisects* CD *at right angles.*

Proof. Let N be the middle point of CD.

Join AN, BN.

Because in the ⊙PQH, B is the centre and N is the mid-point of the chord CD,

∴ BN is ⊥ to CD. *Th. 33.*

Similarly, AN is ⊥ to CD.

∴ the ∠BNC + the ∠ANC = 2 rt. ∠ˢ.

∴ AN and BN are in one and the same str. line.

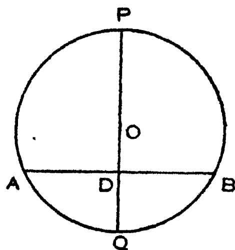
Hence, the str. line AB bisects CD and is also perp. to it.

Q. E. D.

Otherwise : The proposition may also be proved as follows : Through N, the mid-pt. of CD, draw the str. line KNE perpendicular to CD. Then, since the pt. A is equidistant from C and D, it *must lie* on the str. line KE (Th. 25, Bk. I.); and, for a similar reason, the pt. B must lie on KE. Hence the str. line AB coincides with the str. line KE, and ∴ it bisects CD at right angles. Q. E. D.

(2) *A circle is symmetrical about one of its diameters.*

Let PQ be any diameter of the circle APB, of which O is the centre.



It is required to prove that the circle is symmetrical about PQ.

Take any point A on the circumference and draw AD perpendicular to PQ and produce it to meet the circle again at B.

Proof. Because OD is perpendicular to the chord AB.

$\therefore$ AD = DB.

Hence, the diameter PQ bisects AB at right angles. Then, if the figure is folded about PQ. DA falls *along* DB, since $\angle$ODA = $\angle$ODB) and $\therefore$ A on B (since AD = DB); $\therefore$ *any point* A on one semi-circumference will fall on some point B on the other semi-circumference;

$\therefore$ the two semi-circumferences coincide.

Hence, the circle is *symmetrical* about PQ.

Q. E. D.

NOTE. In the above figure, evidently A and B are symmetrically opposite points with regard to PQ. Hence,

if a circle passes through any point A, *it must also pass through the symmetrically opposite point with regard to any diameter.*

EXERCISE 1

1. AB is a chord of a circle of which the centre is O. Prove that the straight line drawn from O to the middle point of AB passes through the middle point of any other chord that is parallel to AB.

2. Prove that in a circle the centre and the middle points of any number of chords that are parallel to one another lie in one straight line.

3. If two chords of a circle bisect each other, prove that their point of intersection must be the centre of the circle.

4. If two circles, whose centres are A and B, have one point P in common, prove that they have also another point Q in common, where Q is such that PQ is bisected at right angles by AB.

5. If any number of circles pass through two given points, prove that their centres all lie in one straight line.

6. If the chords AB, AC of a circle are equally inclined to the radius to A, prove that AB = AC.

7. If a circle pass through the vertices of a right-angled triangle, show that its centre is the middle point of the hypotenuse.

8. If two circles have two points in common, prove that any two parallel straight lines drawn through the common points and terminated by the circumferences are equal.

9. From a point, O, two equal straight lines OA, OB are drawn to the circumference of a circle whose centre is C. Prove that the ∠AOC = the ∠BOC.

10. The radii of two intersecting circles are r and r', and d is the distance between their centres. Show that $r - r' < d < r + r'$.

11. Show how to draw the chord of a circle which is bisected at a given point within the circle.

12. A chord PQ of a circle cuts a concentric circle in P′, Q′; prove that PP′ = QQ′.

13. *Diameter of a circle is the greatest chord.*

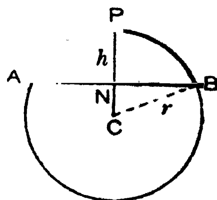
14. In the adjoining figure, APB represents a bridge, of the form of an arc of a circle whose radius is r; if the *height of the arch* be h, find the *span* AB.

[Evidently CP = CB = r;

∴ CN = $r - h$; hence, since the △CNB is rt. ∠*ed* at N,

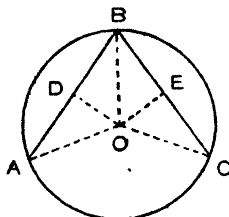
$$BN^2 = CB^2 - CN^2$$
$$= r^2 - (r-h)^2$$
$$= 2rh - h^2.$$

∴ $AB = 2BN = 2\sqrt{2rh - h^2}$.]



THEOREM 34.

There is one circle, and one only, which passes through three given points, not in a straight line.



Let A, B, C be three given pts. which do not lie in one str. line.

It is required to prove that one circle, and only one, passes through A, B *and* C.

Join AB, BC.

Let DO and EO be the perp. bisectors of AB and BC respectively.

Then, ∵ AB and BC are not in one and the same str. line, DO and EO are not parallel;

∴ DO and EO must meet in some point O.

Join OA, OB, OC.

Proof. Because O is on the perp. bisector of AB, we have,

OA = OB.

Similarly, because O is on the perp. bisector of BC, OB = OC.

∴ OA = OB = OC.

Hence, the circle described with centre O and radius OA passes through A, B and C.

Again, because the two str. lines, DO and EO, cannot intersect at any other point, there cannot be *any other point* equidistant from A, B and C.

∴ there is no other circle which passes through A, B and C.

Q. E. D.

COR. 1. *Circles passing through three given points which are not in the same str. line coincide.*

For, they have the same centre and the radius.

COR. 2. *One circle cannot cut another in more than two points.*

For, if one ⊙ has three points in common with another, then there will be *two* ⊙ˢ passing through the *same* three points, which is impossible.

COR. 3. *If from a point* O *within a circle three equal straight lines* OP, OQ, OR *can be drawn to the circumference, then* O *must be the centre of the circle.*

Since the pt. O is equidistant from the pts. P, Q, R, it must be the pt. of intersection of the ⊥ bisectors of PQ and QR; and there is *no other* pt. besides this which can be equidistant from those three pts. Hence the centre of the given ⊙, which certainly is equidistant from the pts. P, Q, R, on the circumference, must coincide with O.

12. Definition : A circle is said to be **described (or circumscribed) about a triangle** when it passes through the vertices of the triangle. The circle described about a triangle is called the **circum-circle** of the triangle; and the centre and the radius of this circle are respectively called the **circum-centre** and the **circum-radius** of the triangle.

The *circum-centre* of a triangle is evidently the point of intersection of the perpendicular bisectors of any two of its sides; and the *circum-radius* is the distance of any vertex of the triangle from the circum-centre.

13. It is to be carefully noted that through any three points not lying on a str. line, a circle may be drawn. But it is only under special conditions that a circle can be drawn through more than three points.

Definition : If a circle can be drawn through four or more points, then those points are said to be **concyclic.**

EXERCISE 2

1. Show that through two given points any number of circles may be drawn, whose centres all lie on a straight line.

2. Is there any flaw in the statement :—

"Through any three given points one circle, and only one, can be described"?

If so, what's the correct statement and why?

3. Describe a circle of given radius through two given points. Is this always possible?

4. Draw a triangle whose sides are 3″, 4″ and 5″; measure the length of its circum-radius.

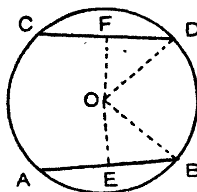
5. Prove that the centre of the circum-circle of an equilateral triangle is equidistant from the sides.

6. Can a circle be *always* drawn through two points, having its centre lying on a given straight line?

THEOREM 35.

Equal chords of a circle are equidistant from the centre.

Conversely, chords which are equidistant from the centre are equal.



Let AB and CD be two chords of a ⊙ whose centre is O.

Let OE, OF be the ⊥ˢ from O upon AB and CD respectively; then OE, OF are respectively the *distances* of AB and CD from O.

If AB = CD, *it is required to prove that* OE = OF.

Join OB, OD.

Proof. Since OE is ⊥ to AB,

∴ EB = $\frac{1}{2}$AB, ∵ AE = EB. *Th. 33.*

Similarly, FD = $\frac{1}{2}$CD.

Hence, EB = FD, ∵ AB = CD. *Hyp.*

Now, in the rt. angled Δˢ OEB, OFD,

∵ {the hypotenuse OB = the hypotenuse OD, and EB = FD;

∴ the two Δˢ are congruent.

Hence, OE = OF.

Q. E. D.

Conversely, if OE = OF, *it is required to prove that*

AB = CD.

Join OB, OD.

Proof. In the rt. angled Δ^s OEB, OFD,

∵ {the hypotenuse OB = the hypotenuse OD. and OE = OF ;

∴ the two Δ^s are congruent.

Hence, EB = FD.

Now, OE being ⊥ to AB, E is the mid-pt. of AB ;

∴ AB = 2EB.

Similarly, CD = 2FD.

Hence, AB = CD, ∵ EB = FD. Q. E. D.

COR. 1. *In equal circles, equal chords are equidistant from the centre, and conversely.*

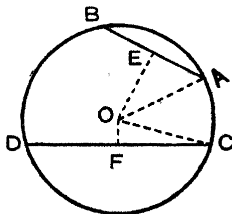
COR. 2. *The middle points of equal chords of a circle lie on the circumference of a concentric circle.*

For, the distances of the centre from the middle points of the chords are equal to one another.

An Important Proposition :

Of any two chords of a circle, the one which is nearer to the centre is greater than the one more remote.

Conversely, the greater of the two chords of a circle is nearer to the centre than the one which is less.



Let AB and CD be two chords of a circle whose centre is O ; and let OE and OF be perp^s from O on AB and CD respectively.

If OF < OE, *it is required to prove that* CD > AB.

Join OA, OC.

Proof. Because OE, OF are perp^s on AB, CD,

∴ AE = $\frac{1}{2}$AB, and CF = $\frac{1}{2}$ CD.

Again, ∵ the ΔOEA is rt. $\angle^{ed}$ at E, ∴ $OA^2 = AE^2 + OE^2$.

Similarly, ∵ the ΔOFC is rt. $\angle^{ed}$ at F,

$OC^2 = CF^2 + OF^2$.

But $OC = OA$, being radii;

$\therefore \quad AE^2 + OE^2 = CF^2 + OF^2$ ....(1)

But $OF < OE$; *i.e.* $OF^2 < OE^2$.

$\therefore \quad CF^2 > AE^2$;

$\therefore \quad CF > AE$; *i.e.* $CD > AB$.

Conversely, if $OF > OE$, *it is reqd. to prove that* $CD < AB$.

Proof. Since $OF > OE$,

$\therefore \quad OF^2 > OE^2$.

$\therefore$ from (1), $CF^2 < AE^2$, *i.e.* $CF < AE$.

$\therefore \quad CD < AB$. Q. E. D.

COR. *Of all chords in a circle, those which pass through the centre are the greatest.*

EXERCISE 3

1. Draw a circle of radius equal to 5 cm. and in it place two chords, each of length 8 cm. Verify by actual measurement that they are equidistant from the centre of the circle.

2. A chord of length 6″ is at a distance of 4″ from the centre of a circle. Find the length of the chord whose distance is 3″ from the centre.

3. O is the centre of a circle of which the radius is 2″ and A, B, C are three points each at a distance of one inch from O. Construct the chords that are bisected respectively at A, B, C and prove that these chords are equal. Verify this by actual measurement.

4. If two equal chords of a circle intersect at a point, they are equally inclined to the diameter of the circle through that point.

(The pt. of intersection may be inside, on or outside the circle).

5. Any two intersecting chords of a circle, equally inclined to the diameter through the point of intersection, are equal.

6. Show how to draw two equal chords of a circle, at right angles to each other.

7. Parallel chords drawn through the extremities of a diameter on opposite sides of it are equal, and conversely.

8. If one chord of a circle passes through the middle point of another chord, show that the former is greater than the latter.

9. Show that there cannot be three equal chords of a circle all passing through a given point within it.

10. AB, AC, AD are three chords of a circle taken in order, AB being the diameter. Prove that AB is greater than AC and AC is greater than AD.

THEOREM 36.

The angle at the centre of a circle is double of an angle at the circumference, standing on the same arc.

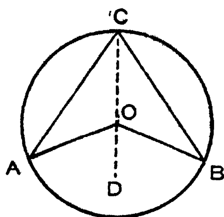


Fig. 1.

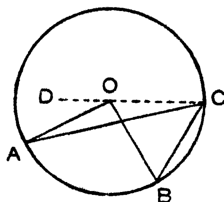


Fig. 2.

Let O be the centre of a ⊙ of which AB is an arc ; and let ∠AOB be the angle at the centre, and ∠ACB an angle at the O^{ce}, standing on the arc AB.

It is required to prove that the ∠AOB = *twice the* ∠ACB

Join CO and produce it to any pt D.

Proof. Since OB = OC, ∵ the ∠OCB = the ∠OBC ;
but the ∠DOB = the ∠OCB + the ∠OBC ;
∴ the ∠DOB = twice the ∠OCB.

Similarly, the ∠AOD = twice the ∠OCA

Hence, in fig. (1),
the ∠AOD + the ∠DOB = twice the ∠OCA + twice the ∠OCB,
i.e. the ∠AOB = twice the ∠ACB ;

and in fig. (2),
the ∠DOB − the ∠AOD = twice the ∠OCB − twice the ∠OCA,
i.e., the ∠AOB = twice the ∠ACB.

Q. E. D.

Obs. If the arc AB be equal to the semi-circumference of the circle, the ∠AOB becomes a *straight-angle* ; and if the arc AB be greater than the semi-circumference, then the ∠AOB becomes a *reflex* angle. The same proof will hold for these two cases too.

EXERCISE 4

1. If O be the circum-centre of a triangle ABC, prove that the angles OBC, BAC are complementary.

2. ABCD is a circle of which the centre is O. If the sum of the angles ADB, BDC be equal to one right-angle, prove that the points A, O, C lie on a straight line.

3. If ABC, A′B′C′ be two triangles in which BC = B′C′, and the ∠A = the ∠A′, show that their circum-circles are equal.

4. Diagonals of a parallelogram, inscribed in a circle, intersect each other at the centre of the circle.

5. Prove that the angle subtended at the circumference of a circle by one of its diameter is a right-angle.

6. From a point A, on the circumference of a circle, AD is drawn perpendicular on a chord BC; if AE be the diameter through A, prove that the angles BAD and EAC are equal.

[Apply Ex. 5.]

7. Show how to determine the positions of three points on a circle such that the triangle formed by joining them has two of its angles equal to 60° and 75°.

8. Show how to divide a circle into two parts, by a chord, such that the angle in one of them subtended by the chord is double that in the other.

[Take any pt. A on the ⊙ whose centre is O and draw a chord AD equal to the radius, r; through O draw OB ∥ to AD. Join OA, DB, so that OADB is a par^{m}. ∴ the ∠ADB = the ∠AOB; take any pt. C on the ⊙, on that side of AB which does not contain D. Then ∠ACB = $\frac{1}{2}$∠AOB and ∴ = $\frac{1}{2}$∠ADB. Hence AB is the chord which divides the ⊙ in the required way.]

CHAPTER III

SEGMENTS : ANGLES IN A SEGMENT

DEFINITIONS

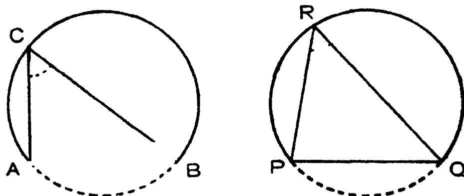
14. A **segment** of a circle is the figure bounded by an arc and the chord joining the extremities of the arc.

NOTE 1. A semi-circle is also a segment.

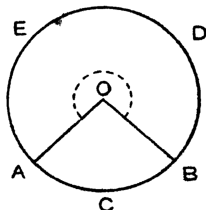
NOTE 2. The chord of a segment is also called the base of the segment.

15. An **angle in a segment** is the angle contained by two straight lines drawn from any point in the arc of the segment to the extremities of its chord.

16. **Similar segments** are those which contain equal angles. Thus, in the following diagram, the segments ACB and PRQ are *similar*, because the ∠ˢ ACB and PRQ are equal.



17. A **sector** of a circle is the figure bounded by an arc and the two radii drawn to the extremities of the arc.



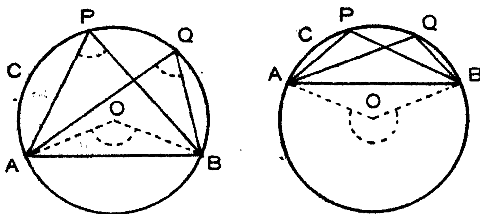
Thus, in the adjoining diagram, the figure AOBC is a sector of the ⊙ABD. The figure AOBDE is also a sector.

NOTE 1. The angle between the two bounding radii of a sector is called the **angle of the sector.** Hence the angle of the sector AOBDE is the re-entrant angle AOB.

NOTE 2. A sector becomes a semi-circle when the angle of the sector becomes equal to two right angles.

THEOREM 37.

Angles in the same segment of a circle are equal.



Let APB be any ∠ in the segment ACB of the ⊙, whose centre is O. Let AQB be *any other* ∠ in the same segment.

It is required to prove that the ∠APB = *the* ∠AQB.

Join OA, OB.

Proof. The arc AB of the ⊙ subtends the ∠AOB at the centre, and the ∠APB at the pt. P on the remaining part of the O^ce ;

∴ the ∠APB = half the ∠AOB. *Th. 36.*

Similarly, the ∠AQB = half the ∠AOB.

Hence, the ∠APB = the ∠AQB.

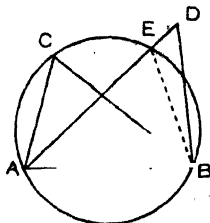
Q. E. D.

NOTE. If the segment ACB be less than a semi-circle, the angle AOB is a *reflex* angle.

THEOREM 38.

[CONVERSE OF THEOREM 37]

If the line joining two given points subtends equal angles at two other points on the same side of it, the four points lie on a circle.



Let C and D be two pts. on the same side of the str. line joining the given points A, B and let the ∠ ACB be = the ∠ ADB.

It is required to prove that the four points A, B, C, D *lie on the circumference of a circle.*

Proof. A ⊙ will pass through *any three* of the four pts. A, B, C, D. *Th. 34.*

∴ if the ⊙ passing through A, B, C do not pass through D, then it will cut AD, or AD produced, at some pt. E.

Join EB.

Then, the ∠ACB = the ∠AEB, in the same segment.

But the ∠ACB = the ∠ADB; *Hyp.*

∴ the ∠AEB = the ∠ADB;

i.e. an ext. ∠ of the △BED is equal to one of its int. opposite angles; which is impossible.

Hence, the circle passing through A, B, C must pass also through D. Q. E. D.

COR. *Triangles standing on the same base, and on the same side of it, with equal vertical angles, have their vertices on the circumference of a circle of which the given base is a chord.*

Let AB be the common base and C, D, E, F, &c., the vertices of the △ˢ; then, since the ∠ACB = the ∠ADB, the ⊙ passing

through A, B, C also passes through D. Similarly, the ⊙ passing through A, B, C also passes through E; and so on. Thus the pts. D, E, F, &c., all lie on the ⊙ that passes through the pts. A, B, C; which proves the corollary.

NOTE. The foregoing corollary and Theorem 37 are converse propositions. For, the corollary may be stated in a slightly altered form, thus :—

If triangles standing on the same base and on the same side of it, have equal vertical angles, then their vertices lie on the arc of a segment of which the given base is the chord; whilst Theorem 37 may also be enunciated as follows :

If triangles standing on the same base, and on the same side of it, have their vertices on the arc of a segment of which the given base is the chord, their vertical angles are equal.

EXERCISE 5

1. A quadrilateral ABCD is inscribed in a circle whose centre is O. If the diagonals AC, BD intersect in E, prove that the $\angle AOB +$ the $\angle COD = 2\ \angle AEB$.

2. If two chords, AC and BD, of a circle, intersect at right angles, show that the chords AB, CD subtend supplementary angles at the centre.

3. AB, CD are two chords of a circle such that when produced they meet at P. If O be the centre of the circle, prove that the angle APC is equal to half the difference between the angles AOC, BOD.

4. Find the locus of the vertex of a triangle having a given vertical angle and standing on a given base.

5. XY is a given straight line and A, B are two fixed points outside it. Find a point in XY at which AB will subtend a right angle.

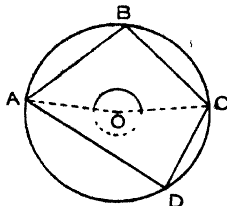
6. Show that the middle points of the sides of a triangle and the feet of the perpendiculars drawn from the vertices to the opposite sides are concyclic.

7. ABCD is a circle. The bisectors of the angles CAB, CBA meet at P, and the bisectors of the angles DAB, DBA meet at Q. Prove that the four points A, Q, P, B are concyclic.

8. ABC is a triangle inscribed in a circle. If AB be fixed in position and C be made to move along the circumference, find the locus of the point of intersection of perpendiculars from A and B on BC and AC, respectively.

THEOREM 39.

The opposite angles of any quadrilateral inscribed in a circle are supplementary.



Let ABCD be a quadrilateral inscribed in a ⊙ whose centre is O.

It is required to prove that (i) the ∠ABC+*the* ∠ADC=*two right angles ;*

and (ii) the ∠BAD+*the* ∠BCD=*two right angles.*

Join OA, OC.

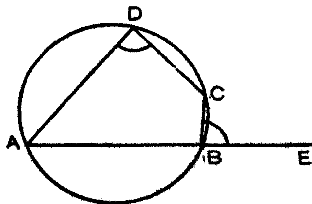
Proof. Now, the ∠ADC at the O^ce=the half the ∠AOC at the centre, standing on the arc ABC.

Also, the ∠ABC at the O^ce
=half the *reflex* ∠AOC at the centre, standing on the arc ADC.

Hence, the ∠ADC+the ∠ABC=half the sum of the ∠AOC and the *reflex* ∠AOC
=half of four right angles
=two right angles.

Similarly, the ∠BAD+the ∠BCD=two right angles. Q.E.D.

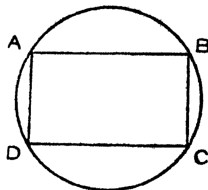
COR. 1. *If the side* AB *of a quadrilateral* ABCD *inscribed in a circle be produced to* E, *the angle* EBC *is equal to the angle* ADC.



The ∠EBC is supplementary to the ∠ABC ; also the ∠ADC is supplementary to the ∠ABC. Hence, the ∠EBC= the ∠ADC.

COR. 2. *If a circle pass through the angular points of a parallelogram, the parallelogram must be a rectangle.*

Let a ⊙ pass through the angular pts. of the par^m ABCD. The ∠A = the ∠C (*opp. ∠^{s} of a par^m*).



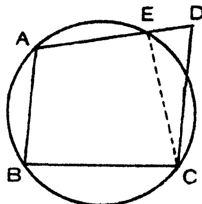
also, the sum of the ∠^{s} A and C is two rt. ∠^{s}, because the quadl. ABCD is inscribed in a ⊙. Hence, each of the ∠^{s} A and C is a rt. ∠, which proves the corollary.

18. Definition. A quadrilateral is said to be **cyclic** when a circle can be made to pass through its angular points.

THEOREM 40.

[CONVERSE OF THEOREM 39]

If a pair of opposite angles of a quadrilateral be supplementary, the quadrilateral is a cyclic one.



Let the ∠^{s} B and D of the quadl. ABCD be together equal to two rt. ∠^{s}.

It is required to prove that the four points A, B, C, D *are concyclic.*

Proof. A ⊙ will pass through the pts. A, B, C. If this ⊙ do not pass through D, let it cut AD, or AD produced, at some pt. E. Join EC.

Now, the quadl. ABCE being cyclic, the ∠AEC is the supplement of the ∠ABC;

but the ∠ADC is the supplement of the ∠ABC. *Hyp.*

∴ the ∠AEC = the ∠ADC, which is impossible.

Hence, the ⊙ which passes through A, B, C must pass also through D. Q. E. D.

COR. *If, when the side* AB *of a quadrilateral* ABCD *is produced to* E, *the angle* EBC *be equal to the angle* ADC, *then the four points* A, B, C, D *are concyclic.*

NOTE. It may be observed that if the ∠ˢ B and D are supplementary the ∠ˢ A and C are also supplementary, because the four angles of a quadrilateral are together equal to four right angles.

EXERCISE 6

1. From a given point, O, outside a given circle, straight lines OAB, OCD are drawn to meet the circle in A, B, C, D respectively. Prove that the triangles OAC, OBD are equiangular.

2. The bisectors of the angles B, C of a triangle ABC meet at P and the bisectors of the exterior angles at B, C meet at Q. Show that B, P, C, Q are concyclic.

3. Any two triangles which stand on equal bases and have their vertical angles supplementary, have equal circum-circles.

4. P is any point in the base AB of an isosceles triangle ABC; prove that the circum-circles of the triangles APC, BPC are equal.

5. ABCD is a quadrilateral inscribed in a circle, the angle ABC is bisected by a straight line BE meeting the circumference at E. If AD be produced to F, show that DE bisects the angle CDF.

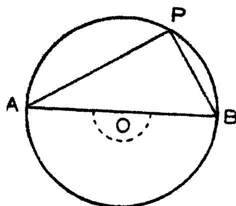
6. ABCD is a quadrilateral and the bisectors of the angles A, B; B, C; C, D; D, A; meet in E, F, G, H respectively. Show that the figure EFGH is cyclic.

7. Prove that a straight line drawn parallel to the base of an isosceles triangle cuts off from it a cyclic quadrilateral.

[Let DE be par to the base BC of an isosceles triangle ABC. Then, ∵ the ∠BDE+the ∠ECB=the ∠BDE+∠the DBC=2 rt. ∠ˢ, the four points B, C, E, D are concyclic.]

THEOREM 41.

The angle in a semi-circle is a right angle.



Let AB be a diameter of a ⊙, of which the centre is O.

Let APB be any ∠ in the semi-⊙ APB.

It is required to prove that the ∠ APB *is a right angle.*

Proof. The semi-○ᶜᵉ AB subtends the *straight angle* AOB at the centre, and the ∠ APB at the pt. P on the remaining part of the ○ᶜᵉ;

∴ the ∠ APB = half the str. ∠ AOB
= half of two rt. ∠ˢ
= one right angle.

Q. E. D.

COR. 1. *Any chord of a circle subtending a right angle at a point on the circumference is a diameter.*

Let AB be the chord subtending a rt. angle at C on the ○ᶜᵉ. If AB be not the diameter, let AD be the diameter through A. Then evidently the ∠ ACD = a rt. ∠ = the ∠ ACB, which is impossible unless AD coincides with AB. Hence the proposition.

COR. 2. *If a straight line* AB *subtends a right angle at any point* C, *then the circle described on* AB *as diameter passes through* C.

If the circle does not pass through C, let it cut AC, or AC produced, at some point D. Join DB. Then the ∠ ADB, being in a semi-circle is a right angle and ∴ equal to the ∠ ACB, which is impossible. Hence, the circle must pass through C.

THEOREM 42.

The angle in a segment greater than a semi-circle is less than a right angle ; and the angle in a segment less than a semi-circle is greater than a right angle.

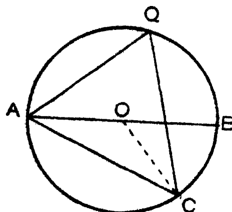


Fig. 1.

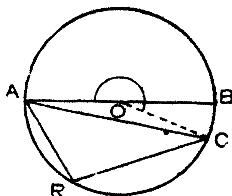


Fig. 2.

(i) Let AC be a chord of the ⊙, whose centre is O, and AB the diameter, as in fig. (1).

Then the segment AQBC is > a semi-⊙ ; let AQC be any angle in this segment.

It is required to prove that the ∠AQC *is less than a right angle.*

Join OC.

Proof. The ∠AQC = half the ∠AOC.

But the ∠AOC is < the ∠AOB *i.e.* < two rt. ∠ˢ;

∴ the ∠AQC is < one right angle.

Q. E. D.

(ii) Let AC be a chord of the ⊙, as in fig. (2).

Then the segment ARC is < a semi-⊙ ;

let ARC be any ∠ in this segment.

It is required to prove that the ∠ARC *is greater than a rt. angle.*

Join OC.

Proof. The arc ABC subtends the *reflex* ∠AOC at the centre and the ∠ARC, at the pt. R on the remaining part of the Oᶜᵉ.

∴ the ∠ARC = half the *reflex* ∠AOC.

But the *reflex* ∠AOC is > two rt. ∠ˢ ;

∴ the ∠ARC is > one right angle. Q. E. D.

EXERCISE 7

1. Two circles intersect at A and B. If AP and AQ are diameters, prove that the three points P, B, Q are collinear.

2. ABC is a triangle, and the circle described on AB as diameter, cuts BC at D. Prove that the circle described on AC as diameter also passes through D.

3. Circles are described on the sides of a quadrilateral as diameters. Prove that the common chord of two of these circles which are adjacent is parallel to the common chord of the other two.

4. If a triangle is inscribed in a circle, prove that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

5. If a quadrilateral is inscribed in a circle, prove that the sum of the angles in the four segments exterior to the quadrilateral is equal to six right angles.

6. ABC is a triangle, and D is the middle point of BC. If BP, CQ be drawn perpendicular to AC, AB respectively, prove that the perpendicular bisector of PQ will pass through D. Prove also that the triangles ABC and APQ are equiangular.

7. In the figure of the preceding example, prove that the angle BCQ is equal to the angle BPQ.

8. Prove that the straight lines, which bisect any angle of a quadrilateral inscribed in a circle and the opposite exterior angle, meet on the circumference of the circle.

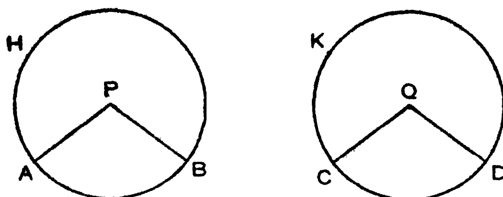
9. A quadrilateral is inscribed in a circle. If the bisectors of two of its opposite angles meet the circle in P and Q, prove that PQ is a diameter.

10. If a trapezium be inscribed in a circle, the oblique sides must be equal.

THEOREM 43.

In equal circles, arcs which subtend equal angles at the centre are equal.

Conversely, in equal circles, angles at the centres, standing on equal arcs, are equal.



Let ABH and CDK be two equal ⊙ˢ whose centres are P and Q; and let the arcs AB and CD subtend the ∠ˢ APB and CQD respectively at the centres.

(*i*) *If the* ∠APB *be equal to the* ∠CQD, *it is required to prove that the arc* AB *is equal to the arc* CD.

Proof. (i) Apply the ⊙CDK to the ⊙ABH, so that the centre Q may coincide with the centre P, and QC may fall along PA.

Then, since the ∠CQD = the ∠APB, QD will fall along PB.

Hence, since QC = PA and QD = PB, the pt. C must coincide with the pt. A and the pt. D with the pt. B.

Also, since the two ⊙ˢ have now become concentric and they have equal radii, the two ○ᶜᵉˢ must coincide *entirely*.

Hence the arc CD entirely coincides with the arc AB and is ∴ *equal* to it.

Q. E. D.

(ii) *Conversely, if the arc* AB *be equal to the arc* CD, *it is required to prove that the* ∠APB=*the* ∠CQD.

Apply the ⊙CDK to the ⊙ABH, so that the centre Q may coincide with the centre P, and QC may fall along PA.

Then, since QC=PA, the pt. C must coincide with the pt. A.

Also, since the ⊙ˢ have now become concentric and have equal radii, the two ○ᶜᵉˢ coincide *entirely*.

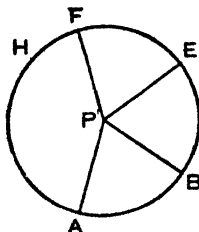
Hence, since the arc CD=the arc AB, the pt. D must coincide with the pt. B.

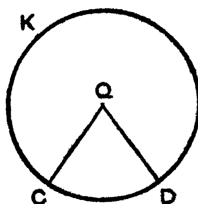
Now, Q coinciding with P and D with B, QD coincides with PB; and consequently the ∠CQD coincides with the ∠APB.

Hence, ∠CQD=the ∠APB.

Q. E. D.

COR. 1. *If in the same circle* ABH, *the arcs* AB *and* EF *subtend equal* ∠ˢAPB, EPF *at the centre, then the arc* AB=*the arc* EF.





Let CDK be a ⊙ whose centre is Q and radius=PA, and let the ∠CQD, contained by the radii QC and QD, be=the ∠EPF (or the ∠APB). Clearly then the arc AB=the arc EF, because each of them=the arc CD.

COR. 2. *If in the same circle* ABH, *the arc* AB *be*=*the arc* EF, *then the* ∠ˢ APB, EPF, *which these arcs subtend at the centre, are equal.*

Let CDK be a ⊙ whose centre is Q and radius = PA, and let the arc CD be = the arc AB (or the arc EF). Clearly then the ∠APB = the ∠EPF, because each of them = the ∠CQD.

COR. 3. *In equal circles, sectors which have equal angles are equal; also sectors which have equal arcs are equal.*

If the ⊙CDK be applied to the ⊙ABH so that the ∠CQD coincides with the ∠APB, then the arc CD also coincides with the arc AB. Hence the sector QCD entirely coincides with the sector PAB and is ∴ equal to it. Similarly, if the arc CD be = the arc AB, it follows that the sector QCD = the sector PAB

COR. 4. *In equal circles (or in the same circle), arcs which subtend equal angles at the circumferences (or circumference) are equal.*

Let O, O′ be the centres of two equal ⊙s. If the arcs AB and CD subtend equal ∠s AEB, CFD at the ○ces, then they also subtend equal ∠s at the centres, (because the ∠s AOB CO′D are respectively doubles of the ∠s AEB, CFD); and ∴ the arc AB = the arc CD.

If in the same ⊙, the arcs AB and GH subtend equal ∠s AEB, GKH at the ○ce, then they also subtend equal ∠s at the centre, (for the ∠s AOB, GOH are respectively doubles of the ∠s AEB, GKH); and ∴ the arc AB = the arc GH.

COR. 5. *In equal circles (or in the same circle), angles at the circumfererces (or circumference) which stand on equal arcs are equal.*

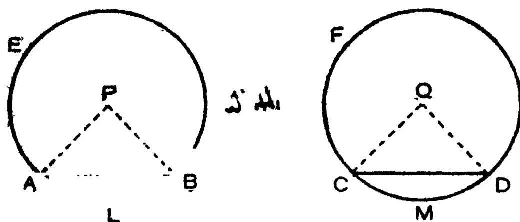
Referring to the notation of Cor. 4, if the arc AB be = the arc CD, then the ∠AOB = the ∠CO′D, (*Th. 43*); and ∴ the ∠AEB = the ∠CFD, because these ∠s are respectively halves of the ∠s AOB, CO′D.

Again, if the arc AB be = the arc GH, then the ∠AOB = the ∠GOH, (*Th. 43, Cor. 2.*); and ∴ the ∠AEB = the ∠GKH, because these ∠s are respectively halves of the ∠s AOB, GOH.

THEOREM 44.

In equal circles, chords which cut off equal arcs are equal.

Conversely, in equal circles, arcs which are cut off by equal chords, are equal, the major arc being equal to the major arc and the minor to the minor.



Let ABE and CDF be two equal ⊙ˢ of which the centres are P and Q.

(i) Let the arc AB be = the arc CD.

Join AB, CD.

It is required to prove that the chord AB *is equal to the chord* CD.

Join PA, PB, QC, QD.

Proof. Because the arc AB = the arc CD,

∴ the ∠APB, standing on the arc AB

= the ∠CQD, standing on the arc CD.

Th. 43.

Hence, in the two Δˢ PAB, QCD,

∵ PA = QC (*radii of equal* ⊙ˢ)

PB = QD („ „ „ „)

and the included ∠APB = the included ∠CQD,

∴ the two Δˢ are congruent.

∴ the chord AB = the chord CD. Q. E. D.

(ii) Conversely, if the chord AB = the chord CD.

It is required to prove that the major arc AEB = *the major arc* CFD *and the minor arc* AB = *the minor arc* CD.

Proof. In the two △ˢ APB, CQD.

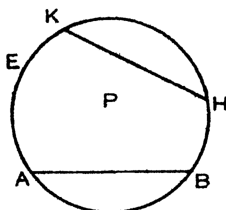
∵ PA = QC (*radii of equal* ⊙ˢ)
PB = QD („ „ „ „)
and AB = CD *Hyp.*

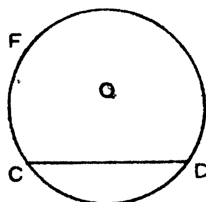
∴ the two △ˢ are congruent.
∴ the ∠APB = the ∠CQD
∴ the arc AB = the arc CD *Th. 43.*
and these are minor arcs.

Again, the ⊙ˢ being equal,
the whole O^{ce} AEBL = the whole O^{ce} CFDM.

∴ the remaining arc AEB = the remaining arc CFD, which proves the equality of the major arcs. Q. E. D.

COR. 1. *If in the same circle* ABE, *the arc* AB *be equal to the arc* HK, *then the chord* AB *is equal to the chord* HK.





Let CDF be an equal ⊙ having its centre at Q, and let the arc CD be = the arc AB (or the arc HK). Clearly, then the chord AB = the chord HK, because each of them = the chord CD.

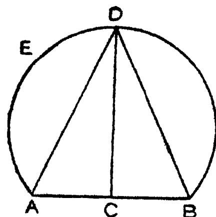
COR. 2. *If in the same circle* ABE, *the chord* AB *be equal to the chord* HK, *then the minor arc* AB *is equal to the minor arc* HK, *and the major arc* AEB *is equal to the major arc* HEK.

Let CDF be an equal ⊙ having its centre at Q, and let the chord CD be = the chord AB (or the chord HK).

Clearly then the minor arc AB = the minor arc HK, because each of them = the minor arc CD. Also the major arc AEB = the major arc HEK, because each of them = the major arc CFD.

COR. 3. *In equal circles, segments which have equal arcs are equal; also, segments which have equal chords and which are both greater, or both less, than a semi-circle, are equal.*

For, if the ⊙CDF be so applied to the ⊙ABE that the arc CD coincides with the arc AB, then the segment CD entirely coincides with the segment AB and is ∴ equal to it. Again, if the chord CD be = the chord AB, the ⊙CDF may be so applied to the ⊙ABE that the arc CD coincides with the arc AB and the arc CFD with the arc AEB. Hence the segment CD entirely coincides with the segment AB and is ∴ equal to it; also the segment CFD entirely coincides with the segment AEB and is ∴ equal to it.



COR 4. *If* C *be the middle point of the chord* AB *of an arc* AEB, *and if* CD *be drawn perpendicular to* AB *meeting the arc in* D, *then* D *is the middle point of the arc* AEB.

Since DC produced is a diameter of the ⊙ of which AEB is an arc, the arcs DA and DB are clearly both less than the semi-○ce of the ⊙. Now, if DA and DB be joined, the two △s ACD and BCD are evidently congruent; whence DA = DB, and ∴ the arc DA = the arc DB.

EXERCISE 8

1. A, B, C are three points on the circumference of a circle of which the centre is O. If the angle AOC be three times the angle AOB, prove that the arc AC is also three times the arc AB.

2. P and Q are two points on the circumference of a circle of which the centre is O. If the ∠POQ = 120°, prove that the arc PQ is one-third of the whole circumference, and that the chord PQ is a side of an equilateral triangle inscribed in the circle.

Hence construct an equilateral triangle with its vertices on the circumference of a circle whose radius is 2 inches.

3. If AB and CD be two diameters of a circle, perpendicular to each other, prove that the quadrilateral ACBD is both equilateral and equiangular.

4. If a regular hexagon be inscribed in a circle, prove that each side of the figure is equal to a radius of the circle.

5. If two equal circles intersect so that the centre of one circle is on the circumference of the other, prove that one-third of each circumference lies within the other.

6. If in a circle, two arcs be such that they subtend complementary angles at the circumference, prove that the two arcs are together equal to half the circumference.

7. If AB, CD be two chords of a circle at right angles to each other prove that the sum of the arcs AC, BD is equal to half the circumference of the circle.

8. AB is the common chord of two equal circles. If any straight line drawn through B meet the circumferences in P and Q, prove that the triangle PAQ is isosceles.

9. The perpendiculars from A, B to the opposite sides of the triangle ABC meet the circum-circle of the triangle at X and Y respectively. Prove that the arc CX = the arc CY.

10. The bisector of the vertical angle A of the triangle ABC bisects the arc BC of the circum-circle of the triangle.

11. OA, OB are two radii of circle at rt. angles to each other, and AX, BY are parallel chords. Prove that BX, AY are rt. $\angle$s to each other.

[Evidently, the Δs BOX, AOY are congruent, and $\therefore$ the $\angle$OBX = the $\angle$OAY.]

CHAPTER IV

TANGENCY

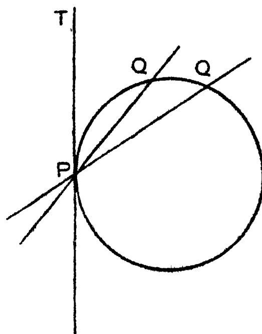
19. Definitions : (i) Any unlimited straight line that cuts the circumference of a circle is called a **secant** of a circle.

Hence, a chord when produced both ways becomes a *secant*, or, in other words, a chord is that portion of a secant which lies within the circle.

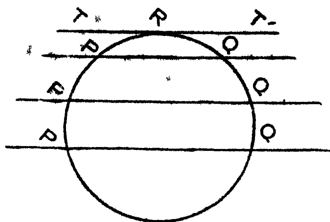
(ii) If a secant of a circle moves so that the two points in which it cuts the circumference continually approach one another, then in the position which it ultimately takes up when the two points coincide, the secant becomes what is called a **tangent** to the circle ; and it is said to *touch*, or be the *tangent* to, the circle *at the point* at which those two points coincide.

(iii) The point at which the tangent to a circle touches it, is called the **point of contact** of the tangent.

Ex. 1. In the adjoining diagram, let PQ be a secant of the circle. Let PQ be so rotated that P remains fixed while Q moves along the circumference towards P ; then the secant PQ will continually change its position. If PT be the position which the secant assumes when Q coincides with P, then PT is the *tangent* to the circle at the point P.



Ex. 2. Let P, Q be any two points on the circumference of a circle. Suppose that the secant PQ moves in such a manner that *both* the points P and Q approach nearer and nearer to some point R on the arc PQ, and ultimately coincide at R. If TRT′ be the position of the secant when P and Q *coincide*, then TRT′ is the *tangent* to the circle at the point R.



NOTE 1. From the above it is clear that when a straight line becomes a *tangent* to a circle at any point, it does not *cut* the circumference at that point. Hence a tangent to a circle is *an unlimited straight line which meets the circumference but does not cut it at the point of meeting.*

NOTE 2. A tangent may also be regarded as a secant cutting the circumference in two *coincident* points. In other words a tangent is a secant of which the chord-portion does not exist.

20. If the circumference of one circle passes through two points on the circumference of another, and if one of the circles be made to move, keeping the other fixed, in such a way that the two points come nearer and nearer to each other, then in its *ultimate position*, when the two points *coincide* into one, the moving circle is said to **touch** the other at the point of coincidence, this point being called their **point of contact.**

For instance, in the diagram p. 192, let O and O′ be the centres of the two circles which cut each other in the points P and Q. If the first circle moves, while the second remains fixed,

so that the points P and Q come nearer and nearer to one another, and if HRK be the ultimate position of the moving circle when P and Q coincide at R, then the circle HRK is said to *touch* the circle RMN at the point R, and R is called their *point of contact.*

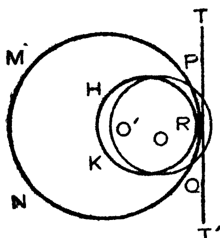


Fig. 1.

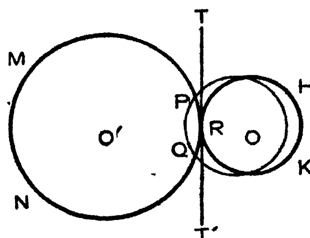


Fig. 2.

Obs. 1. Hence it is clear that two circles may be said to *touch* each other when they *meet* but *do not cut each other* at the point of meeting.

Obs. 2. It may also be said that one circle touches another when the circumference of the former passes through two *coincident* points on the circumference of the latter.

Obs. 3. In the above diagram, let TRT′ be the ultimate position of the secant PQ, when P and Q coincide at R.

Then TRT′ is clearly a tangent to each of the two circles because it passes through two *coincident* points on each circumference. Hence, *when two circles touch each other, they have a common tangent at the point of contact.*

21. Definition : Two circles are said to touch each other **internally** (or to have **internal contact),** when the centres of the circles are on the *same* side of the point of contact, as in fig. (1) ; and they are said to touch each other **externally** (or to have **external contact),** when the centres are on opposite sides of the point of contact, as in fig. (2).

THEOREM 45.

The tangent at any point of a circle is perpendicular to the radius through the point of contact.

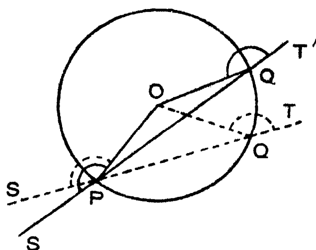


Fig. 1.

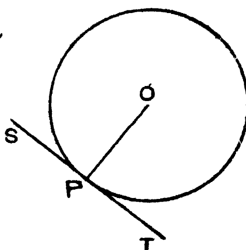


Fig. 2.

Let P be any point on the $\bigcirc^{ce}$ of a ⊙ whose centre is O; join OP.

It is required to prove that the tangent at P *is perpendicular to* OP.

Let ST be a secant through P, cutting the ⊙ again at Q. (Fig. 1).

Join OQ.

Proof. Since OQ = OP, ∴ the ∠OPQ = the ∠OQP.

Now, the ∠OPS is the supplement of the ∠OPQ,
and the ∠OQT is the supplement of the ∠OQP.

Hence the ∠OPS = the ∠OQT.

Let the secant ST be turned about the pt. P so that the pt. Q, moving along the arc QP, gradually approaches P.

Then, in *every position* of the secant, the ∠OPS = the ∠OQT.

Ultimately, when Q coincides with P, the secant ST becomes the tangent at P, and the ∠OQT becomes the ∠OPT, as in fig. 2.

Hence, when ST is the tangent at P, the ∠OPS = the ∠OPT, and these are adjacent angles; ∴ each = a rt. ∠.

∴ OP is ⊥ to ST.

That is, the tangent at P is ⊥ to OP. Q. E. D.

COR. 1. *If from the point of contact of a tangent to a circle a straight line be drawn perpendicular to the tangent, it must pass through the centre of the circle.*

If PO′ be drawn ⊥ to PT, PO′ *must* coincide in direction with PO; for, otherwise there would be two perpendiculars to PT at P, which is impossible.

COR. 2. *If through any point on a circle a straight line be drawn perpendicular to the radius to that point, it must be the tangent to the circle at that point*

If PT′ be drawn ⊥ to OP, PT′ *must* coincide in direction with PT; for, otherwise there would be two perpendiculars to OP at P, which is impossible.

COR. 3. *There cannot be more than one tangent to a circle at any given point on its circumference.*

For, if there were two tangents to a ⊙ at any given pt. P on its ○ᶜᵉ, each of them would be ⊥ to OP, and thus there would be two ⊥ˢ to OP at P, which is impossible.

COR. 4. *If through the centre of a circle a straight line be drawn perpendicular to a tangent to the circle, it must meet the tangent at the point of contact.*

If through O a straight line be drawn ⊥ to the tangent SPT, it *must* coincide in direction with OP; for, there would otherwise be two perpendiculars to SPT through O, which is impossible.

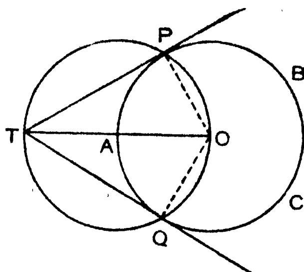
COR. 5. *Every point on a tangent to a circle except the point of contact, is outside the circle.*

Let SPT be a tangent to a ⊙ whose centre is O, the pt. of contact being P. Then, since OP is ⊥ to ST, every other point on ST, besides P, is at a greater distance, from O, than OP, *i.e.*, at a greater distance from O than the radius of the ⊙; which proves the corollary.

NOTE. Theorem 45 may also be proved as follows: The straight line drawn from the centre of a circle to the middle point of the chord-portion of a secant is always perpendicular to the secant (Th. 33). When the secant becomes a tangent, the chord-portion becomes reduced to the mere point of contact, and so the middle point of the chord-portion too becomes identical with the point of contact. Hence the straight line drawn from the centre to the point of contact of a tangent (which may be regarded as the straight line drawn from the centre to the middle point of the chord-portion of the tangent) is perpendicular to the tangent.

THEOREM 45A.

Two, and only two, tangents can be drawn to a circle from an external point.



Let ABC be a ⊙ whose centre is O and T be an external point.

It is required to prove that two tangents can be drawn from T *to the* ⊙ ABC, *and no more.*

Join OT, and on OT as diameter let a ⊙ TPQ be drawn, cutting the ⊙ABC at P and Q.

Join TP, TQ, OP, OQ.

Proof. In the ⊙TPQ, ∠'s TPO, TQO, being in semi-circles, are each a right angle.

∴ TP and TQ are perps. to the radii OP, OQ, respectively, through the points P, Q on the O^{ce} of the ⊙ABC.

∴ TP and TQ are each a tangent to the ⊙ABC.

Again, because a tangent TP is obtained by joining T to a point of intersection of the ⊙s TPQ and ABC, and because two ⊙s cannot intersect in more than two points, it follows that there can be no more than two tangents from T to the ⊙ABC.

COR. *No tangent can be drawn to a circle from a point within it.*

For, if T be a point within the ⊙ABC, the ⊙ on OT as diameter will not cut the ⊙ABC.

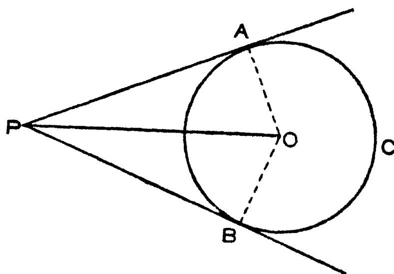
22. Definitions : When a tangent is drawn to a circle from a point outside it, *the portion of the tangent intercepted between the point and the point of contact* is often spoken of as **the tangent drawn from the point** to the circle.

Thus, in the diagram of *Th. 46*, PA, PB are called the *tangents* drawn from P to the circle.

If AB be joined, the line AB is called the **chord of contact** of the tangents from P to the circle.

THEOREM 46.

The two tangents drawn from an external point to a circle are equal and subtend equal angles at the centre.



Let PA and PB be the two tangents from an external point P to a circle ABC, whose centre is O ; and let PA and PB touch the ⨀ABC at A and B respectively. Join OP, OA, OB.

It is required to prove that PA = PB
and *the* ∠POA = *the* ∠POB.

Proof. Because PA, PB are tangents and OA, OB are radii through their points of contact,

the ∠PAO = the ∠PBO, each being a rt. ∠.

Now, in the right $\angle^{ed}$ Δ^s PAO, PBO,

$\because$ { OA = OB, radii of the same ⊙
and the hypotenuse OP is common,

$\therefore$ the two Δ^s are congruent.

$\therefore$ PA = PB

and the $\angle$POA = the $\angle$POB.

Q. E. D.

COR. *The two tangents drawn to a circle from an external point are equally inclined to the diameter through that point.*

EXERCISE 9

1. If two circles are concentric, prove that any chord of the outer circle which touches the inner is bisected at the point of contact.

2. If any number of equal circles touch a given straight line on the same side of it, prove that their centres all lie in one straight line.

3. If P be a point from which tangents are drawn to a circle whose centre is O, prove that OP is the perpendicular bisector of the chord of contact.

4. If any number of circles pass through the same point and touch one another at that point, prove that their centres all lie in one str. line.

5. If any number of circles touch each of two intersecting straight lines, prove that their centres all lie in one straight line.

6. If a parallelogram be described about a circle, prove that the sum of one pair of its opposite sides is equal to that of the other pair. Hence prove that any two adjacent sides of the parallelogram are equal.

7. If a quadrilateral be such that the sum of one pair of its opposite sides is equal to that of the other pair, prove that a circle which touches any three sides of this quadrilateral must also touch the fourth.

•8. If the two tangents, one to each of two intersecting circles at either point of intersection, be at right angles to each other, show that the sum of the squares on their radii is equal to the square on the distance between their centres.

THEOREM 47.

If two circles touch, the point of contact lies in the straight line through the centres.

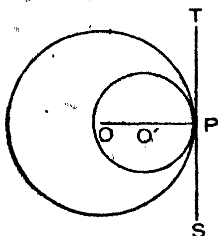


Fig. 1.

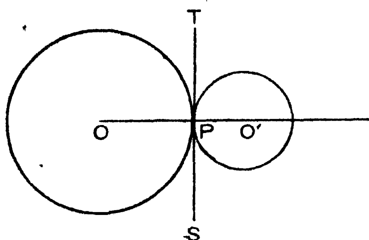


Fig. 2.

Let the ⊙ˢ whose centres are O and O′ touch at the pt. P.

It is required to prove that O, O′ *and* P *lie in the same str. line.*

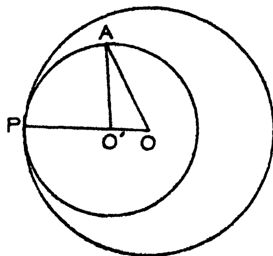
Proof. Because the two circles touch at P they have a common tangent at P.

Let TPS be the common tangent ; join OP, O′P.

Then OP, O′P being radii drawn to the pt. of contact, each of the ∠ˢ TPO, TPO′ is a rt. ∠.

∴ OP and O′P are in one and the same str. line *i.e.* the three pts. O, O′, P lie in one str. line. Q. E. D.

COR. I. *If two circles touch each other internally, every point on the circle of smaller radius, except the point of contact lies inside the other circle.*



Let the ⊙ˢ whose centres are O and O′ touch each other internally at P, the second ⊙ being of smaller radius. Then O, O′, P are in one str. line and OP>O′P. Take *any* pt. A, *other than* the pt. P, on the ⊙ of smaller radius ; join OA, O′A. Now, OA is less than the sum of OO′ and O′A and is ∴ less than OP. Thus the distance of A from the centre of the first ⊙ is less than its radius ; ∴ A is *inside* the first ⊙, which proves the corollary.

COR. 2. *If two circles touch each other externally, every point on either circle, except the point of contact, lies outside the other.*

COR. 3. *The distance between the centres of two circles, which touch one another, is equal to the sum or difference of their radii according as their contact is external or internal.*

NOTE 1. The main proposition may also be proved as follows: If two circles cut each other in two points, their centres and the mid-pt. of the common chord lie in one str. line. When the circles touch, the two pts. of intersection coincide; the common chord, therefore, becomes reduced merely to the pt. of contact, and consequently the mid-pt. of the chord also becomes identical with the point of contact. Hence, when the circles touch, their centres and the point of contact must lie in one str. line.

NOTE 2. *If two circles have a common point on the line joining their centres, the circles touch each other at the point.*

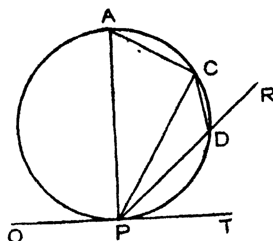
For, if the circles intersect in two distinct points, the line joining the centres passes through the mid-pt. of the common chord and never through one or other of the pts. of intersection.

NOTE 3. *If two circles have the same tangent at a common point they touch each other at the point.*

For, if the circles intersect in two distinct pts., they cannot have a common tangent at either of those pts.; they can only have a common *secant* passing through the two pts. of intersection.

THEOREM 48.

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in alternate segments respectively.



Let QPT be the tangent, touching the ⊙ACP at P, and PC be any chord which divides the ⊙ into two segments CAP and CDP.

It is required to prove that

(*i*) *the* ∠CPT = *any* ∠ *in the segment* CAP.

(*ii*) *and the* ∠CPQ = *any* ∠ *in the segment* CDP.

Let PA be the diameter through P. Join AC, CD, PD (D being any point in the segment CDP).

Proof. (i) ∵ PA is the diameter through the pt. of contact, the ∠APT = a rt. ∠.

i.e. the ∠APC + the ∠CPT = a rt. ∠................(1)

Also, the ∠PCA, being in semi-⊙, = a rt. ∠.

∴ the ∠APC + the ∠PAC = a rt. ∠................(2)

from (1) and (2), taking away the common ∠APC, we have,

the ∠CPT = the ∠PAC = any ∠ in the segment CAP, standing on the arc CP.

(ii) Again, PACD is a cyclic quadril.;
∴ the ∠PAC + the ∠CDP = 2 rt. ∠s;
also, the ∠CPT + the ∠CPQ = 2 rt. ∠s.

But, the ∠CPT = the ∠PAC *proved*
∴ the ∠CPQ = the ∠CDP = any ∠ in the segment CDP.
Q. E. D.

Proof by the method of Limits.

Proof. In the arc CDP, taken a point D close to P; join PD and produce it to any pt. R. Join CD, AC, AP.

Now, the quadl. PDCA being cyclic, the ∠CDR = the ∠CAP; *Th. 39, Cor. 1.*

and this equality holds good *however close* the point D may be to P.

Suppose the secant PR to be turned about P, so that D moves towards P; then, ultimately, when D coincides with P, the ∠CDR becomes the ∠CPT.

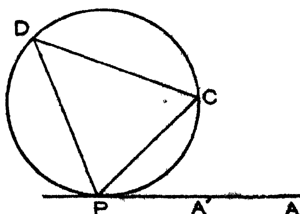
Hence the ∠CPT = the ∠CAP.

By similar construction and reasoning, the ∠CPQ = the ∠CDP. Q. E. D.

COR. *If* PDC *is a triangle, and if through* P *a straight line* PA *be drawn on the side of* PC *remote from* D *such that the angle* CPA *is equal to the angle* CDP, *then* PA *is the tangent at* P *to the circum-circle of the triangle* PDC.

If PA′ be the tangent at P, to the circum-circle, then the ∠CPA′ = the ∠CDP; but the ∠CDP = the ∠CPA. *Hyp.*

∴ the ∠CPA′ = the ∠CPA; hence PA′ and PA coincide, *i.e.*, PA is the tangent at P to the circle PDC.



EXERCISE 10

1. P is the middle point of an arc APB of a circle. Prove that the tangent at P is parallel to the chord AB.

2. If an equilateral triangle be inscribed in a circle, prove that the tangents at its angular points will form another equilateral triangle.

3. Two circles intersect at A and B ; and through P, any point on the circumference of one of them, str. lines PAC, PBD are drawn to cut the other ⊙ at C and D. Prove that CD is ∥ to the tangent at P.

4. ABC is a triangle right-angled at C ; and from C a perpendicular CD is drawn to the hypotenuse. Prove that BC bisects the angle between CD and the tangent at C to the circum-circle of the △ABC.

5. Construct a triangle, having given the base, the vertical angle, and the length of the str. line joining the vertex to the middle pt. of the base.

6. Two ⊙s touch each other externally. Prove that any str. line drawn through the pt. of contact cuts off similar segments from the two ⊙s.

7. ABC is a triangle, and DE is drawn parallel to BC cutting the sides in D and E. Prove that the circles circumscribed about the triangles ADE and ABC have a common tangent at A.

MISCELLANEOUS EXERCISES

(*On Circles*)

1. Any number of equal circles pass through a given point ; show that their centres all lie on a fixed circle.

2. Show that all circles, passing through one fixed point and having their centres lying on a fixed straight line, pass through a second fixed point.

3. If two circles intersect, prove that their common chord produced bisects their common tangent.

4. Find the locus of the middle points of equal chords in a given circle.

5. Show how to place in a given circle a chord of given length parallel to a given str. line.

6. Find the chord of minimum length through a given point within a given circle.

7. Find the locus of the centres of circles which are touched by two given straight lines.

8. Of all chords drawn through a given point within a circle, the least is that which is bisected at that point.

9. A straight line drawn through one of the points of intersection of two given circles is terminated by the circumferences ; show that the angle between the tangents at its extermities is equal to that between the tangents at the point of intersection.

10. Any two chords drawn, one in each, of two equal intersecting circles, through one of their points of intersection and equally inclined to the common chord are equal.

11. Two circles touch each other internally at P ; if AQ and BR be two parallel radii of the two circles drawn in the same sense, show that P, Q, R are in the same str. line.

12. Two equal circles touch one another externally, and two chords are drawn to the two circles through their point of contact at right angles to each other ; show that the straight line joining the remaining extremities of the chords is equal to the diameter of either of them.

13. Parallel chords of a circle cut off equal arcs between them and conversely.

14. The oblique sides of a trapezium inscribed in a circle are equal.

15. Two circles intersect at A and B and O is any point on one of them ; if OA, OB be produced to cut the other circle in K and L, show that KL is of constant length.

16. Two equal segments of circles are described on opposite sides of the same chord AB ; and through O, the middle point of AB, any str. line POQ is drawn intersecting the arcs of the segments at P and Q ; show that OP=OQ.

17. Circles are described on the sides of a quadrilateral as diameters ; show that the common chord of any two consecutive circles is parallel to the common chord of the other two.

18. Find the locus of the middle points of chords of a circle, drawn through an external point.

19. If the sum of one pair of opposite sides of a quadrilateral be equal to the sum of the other pair, show that a circle may be inscribed within it.

20. If from an internal point, other than the centre, straight lines are drawn to the circumference of a circle, then the greatest is that which passes through the centre, and the least is the remaining part of that diameter.

And of any other two such lines the greater is that which subtends the greater angle at the centre.

21. If from an external point straight lines are drawn to the circumference of a circle, the greatest is that which passes through the centre, and the least is that which when produced passes through the centre.

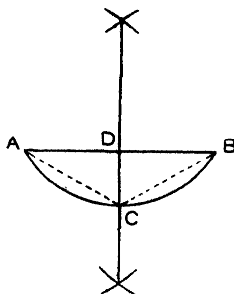
And of any other two such lines, the greater is that which subtends the greater angle at the centre.

CHAPTER V

PROBLEMS ON BISECTION OF ARCS AND CONSTRUCTION OF CIRCLES

PROBLEM 24.

To bisect a given arc of a circle.



Let ACB be an arc of a circle.

It is required to bisect the arc ABC.

Construction. Join AB and bisect it at right angles by DC, meeting the arc at C.

Then C is the required point of bisection.

Proof. Join AC, BC.

Because C is a point on the perp. bisector of AB,

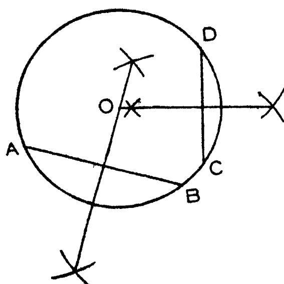
$\therefore$ CA = CB.

Hence, $\because$ the chord CA = the chord CB

$\therefore$ the arc CA = the arc CB. Q. E. F.

PROBLEM 25.

Given a circle, or an arc of a circle, to find the position of the centre.



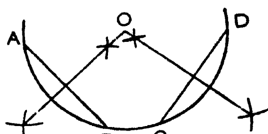


Fig. 1. Fig. 2.

Let ABC be a circle, (fig. 1), or an arc of a circle (fig. 2).

It is required to find the position of the centre.

Construction. Let AB, CD be any two chords of the circle, or the arc.

Draw the perp. bisectors of AB and CD, and let them intersect at O.

Then O is the required centre.

Proof. Because A and B are points on the circle or the arc, the centre is equidistant from A and B.

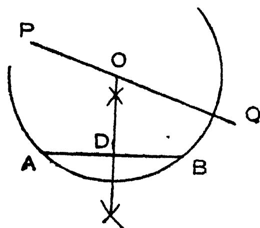
∴ it lies on the perp. bisector of AB.

Similarly, the centre lies on the perp. bisector of CD.

∴ the centre is at O, the point common to these perp. bisectors. Q. E. F.

PROBLEM 26.

To construct a circle which passes through two given points and has its centre lying on a given straight line.



Let A, B be the given points and PQ, the given straight line.

It is required to draw the circle passing through A, B *and having its centre lying on* PQ.

Construction. Join AB and bisect it at right angles by OD.

Produce DO, to meet PQ at O.

Draw the circle with O as centre and OA as radius.

This is the required circle.

Proof. ∵ OD is the perp. bisector of AB, any point on OD is equidistant from A and B.

∴ the centre of the required circle, being equidistant from A, B, must lie on OD.

Also, the centre of the required circle lies on PQ.

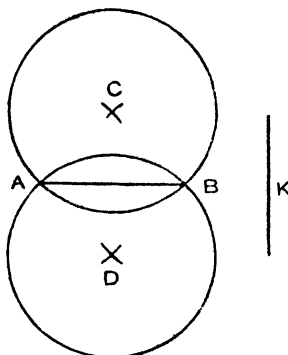
Hence, the centre of the circle, to be constructed, is at O, the point common to DO and PQ.

∴ the circle with O as centre and OA as radius is the required one. Q. E. F.

Remark: It should be observed that the construction will *fail* if the given straight line, PQ, be perpendicular to that joining the given points, A, B, for in that case, PQ will be parallel to OD and will not therefore intersect.

PROBLEM 27.

To construct a circle passing through two given points and having a given radius.



Let A and B be the given points, and K represent the length of the given radius.

It is required to draw a circle, through A, B *having its radius equal to* K.

Construction. With centre A and radius 'K' draw a circle; also, with centre B and radius 'K', draw another circle cutting the former in C and D.

Then, the circle, with centre C and radius CA, is the required circle.

Also, the circle with centre D and radius DA, is another circle satisfying the given conditions.

Thus, there are two solutions of the given problem.

Proof. ∵ CA = K and CB = K,

∴ CA = CB, each being = K.

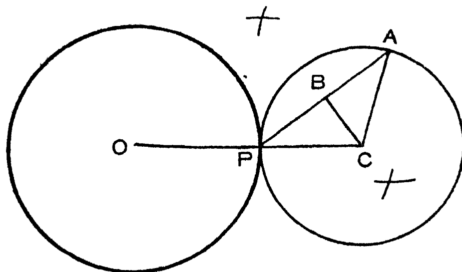
∴ the circle whose centre is C and radius CA, *i.e.* K, passes through A, B.

Similarly, the circle with centre D and radius DA *i.e.* K, passes through A, B. Q. E. F.

Remark: The construction *fails* if the given radius be less than half the distance between the given points, for, in that case, the two circles described with centres A, B will not intersect.

PROBLEM 28.

To construct a circle passing through a given point, outside a given circle, and touching the given circle at a given point.



Let A be the given pt. outside the given ⊙ whose centre is O, and let P be the given pt. on the O^{ce} of the given ⊙.

It is required to describe a circle passing through A *and touching the given circle at* P.

Construction. Join AP; and draw the ⊥ bisector of AP meeting OP produced in C.

Then the ⊙ described with C as centre and CP as radius, will be the reqd. ⊙.

Proof. Join CA.

BC being the ⊥ bisector of AP, CA = CP;

and ∴ the ⊙ described with C as centre and CP as radius, will also pass through A.

Also, since the two ⊙ˢ meet at P which is a pt. on the str. line joining their centres, ∴ the ⊙ˢ touch each other at P.

Hence, the ⊙ described with C as centre and CP as radius is the reqd. ⊙. Q. E. F.

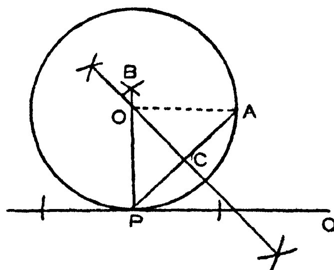
Analysis: The solution of the present problem is guessed in the following manner: *Assume* a circle to be described passing through A and also touching the given circle at P, and let C be its centre. Then, since the circles touch each other at P, OP produced must pass through C; also, since CA = CP, C must be on the perpendicular

bisector of AP. Hence C is the pt. where the perpendicular bisector of AP meets OP produced.

The process by which the solution of a problem is discovered is called **analysis**; whilst the opposite process, that of giving the necessary construction and proving its correctness, is called **synthesis.**

PROBLEM 29.

To construct a circle which is touched by a given straight line at a given point and passes through another point outside str. line.



Let P be the point on the given str. line PQ and A, another point lying outside PQ.

It is required to draw a circle which passes through A *and is touched by* PQ at P.

Construction. Through P, draw PB ⊥ to PQ.

Also, join PA and bisect it at right angles by CO. Let CO and PB intersect at O.

Draw the circle with centre O and radius OP.

This is the required circle.

Proof. ∵ O is a point of the perp. bisector of PA,

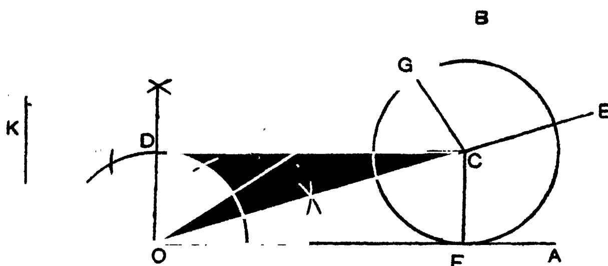
OP = OA.

∴ the circle with O as centre and OP as radius passes through P and A.

Also, ∵ PQ is ⊥ to the radius OP through its extremity, P, on the O^{ce}, PQ touches the circle at P. Q. E. F.

PROBLEM 30.

To construct a circle, with a given radius, touching two given intersecting straight lines.



Let OA, OB be the two given str. lines intersecting at O, and let K be the given radius.

It is required to describe a circle which shall touch OA *and* OB, *and have its radius equal to* K.

Construction. Draw OE, the bisector of the ∠ AOB.

Draw OD ⊥ to OA, making it = K.

Through D, draw a str. line ∥ to OA meeting OE in C.

Draw CF, CG ⊥ to OA, OB respectively. Then the ⊙ described with C as centre and CF as radius will be the reqd. ⊙.

Proof. C being on the bisector of the ∠ AOB, CF = CG.

∴ the ⊙ described with C as centre and CF as radius will also pass through G.

Since OA is ⊥ to the radius CF at F,

∴ OA touches the ⊙ at F.

Similarly, OB touches the ⊙ at G.

Thus, OA and OB are both tangents to the ⊙............(1)

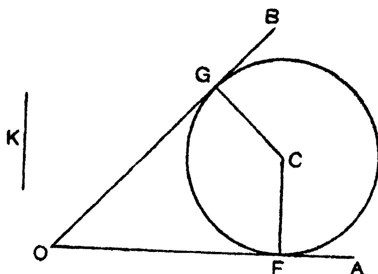
Again, since DF is a rectangle, by construction,

∴ CF = DO = K ....(2)

Hence, from (1) and (2), the circle described with C as centre and CF as radius is the reqd. ⊙. Q. E. F.

NOTE. The following is the *analysis* of the problem :—

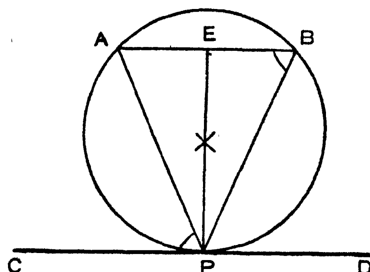
Assume a ⊙ to be described touching OA and OB at F and G respectively, and having its radius = K; and let C be the centre of the ⊙. Join CF, CG. Then CF and CG, being radii drawn to the pts of contact, are respectively ⊥ to OA and OB. Now, since CF = CG ∴ C lies on the bisector of the ∠ AOB;



and also since CF = K, ∴ C lies on a str. line which is ∥ to OA and at a distance = K from it. Hence, if these two lines be drawn, the pt. where they intersect will be the centre of the reqd. circle.

PROBLEM 31.

To construct a circle passing through two given points, and touching a given straight line, parallel to that joining the given points.



Let A, B be the two given pts. and CD the given str. line.

It is required to describe a circle passing through A *and* B, *and also touching the str. line* CD.

Join AB. Then AB is ∥ to CD.

Construction. Draw the ⊥ bisector of AB meeting CD in P.

Then the ⊙ passing through the pts. A, B, P will be the reqd. ⊙.

Proof. Join PA, PB.

Now, P being on the ⊥ bisector of AB, PA = PB;
and ∴ the ∠PBA = the ∠PAB.

Since AB is ∥ to CD, ∴ the ∠CPA = the ∠PAB.

Hence, the ∠CPA = the ∠PBA, in the alt. segment;
∴ CD touches the ⊙ at P.
Hence, ABP is the reqd. ⊙. Q. E. F.

Remarks on the Construction of Circles

23. From the definition of a circle, it is clear that a circle can be described whenever we know the position of its *centre* and the *length* of its *radius*. Now, the centre is determined, as we have seen in the foregoing constructions, as *the point* of intersection of two locii *e.g.* the point of intersection of two straight lines, or, the points of intersection of two circles, or, the points of intersection of a straight line and a circle etc., and as such, requires *two* given *conditions* for its determination. When the centre is thus found out, the *radius* is obtained as the *distance* of the centre from either another given point or, another straight line, or, as the *distance* between two parallel straight lines etc. and therefore requires another condition to be fulfilled.

Hence, *three independent* conditions must always be given before a circle may be described. As for example.

(1) *To describe a circle through the vertices of a given triangle ;*

[The centre is the pt· of intersection of the perp. bisectors of any two sides and the radius is the distance of the centre from any of the vertices].

(2) *to describe a circle touching the sides of a triangle ;*

[The centre is the pt. of intersection of the bisectors of the angles and the radius is the distance of the centre from any of its sides.]

(3) to describe a circle through two given points, having its radius equal to a given length ; and so on. *See problem 27.*

It is however found that in some cases, more than one solution comes out of the same given conditions ; but in that case, too, it is seen that there is some ***intermediate*** solution under which the different solutions obtained combine into a *single* one.

24. To facilitate construction of problems, it is desirable that the students should be quite familiar with the following locii :—

(*1*) ***The locus of the centres of circles passing through two fixed points.***

(2) *The locus of the centres of circles touching two (i) intersecting straight lines, (ii) parallel straight lines.*

(3) *The locus of the centres of circles touching (i) a given straight line at a given point, (ii) touching a given circle at a given point.*

(4) *The locus of the centres of circles, having a given radius and touching (i) a given straight line, (ii) a given circle.*

(5) *The locus of the middle points of chords of a circle. (i) which pass through a given point, (ii) which are of constant length, (iii) which are parallel.*

(6) *The locus of the vertices of a triangle standing on a given base and having a given vertical angle.*

EXERCISE 11

(On Construction of Circles.)

1. Draw a circle through two given points A, B and having its centre at a distance of 2″ from AB.

2. Describe a circle through two given points, having its centre lying on a given circle of radius 3″. How many solutions are there ?

3. Construct a circle through the points of intersection of two given circles and having its centre lying on a given straight line.

4. Describe a circle touching two given parallel straight lines, and passing through a given point within those lines.

5. Describe a circle touching each of three given straight lines, two of which are parallel and the other intersecting them.

6. Show how to draw a circle touching each of three given straight lines which intersect each other, two by two. How many solutions are possible ?

7. Construct a circle touching a given circle and having its centre at a given point.

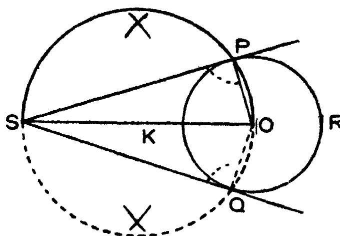
8 Describe a circle whose centre lies on a given straight line and which touches a given circle at a given point on it.

9. Draw a circle of given radius to touch a given circle and to pass through a given point outside the given circle.

10. Describe a circle of given radius touching two given intersecting circles. How many solutions are there ?

PROBLEM 32.

To draw a tangent to a given circle from a given external point.



Let PQR be the given circle whose centre is O, and S, the given external point.

It is required to draw a tangent from S *to the circle* PQR.

Construction. Join SO and bisect it at K. With K as centre and KO as radius draw a circle to cut the ⊙PQR, at P.

Join SP.

Then SP is the required tangent.

Proof. Join PO.

Evidently, SO is the diameter of the ⊙SPO,

∴ the ∠SPO = a rt. ∠.

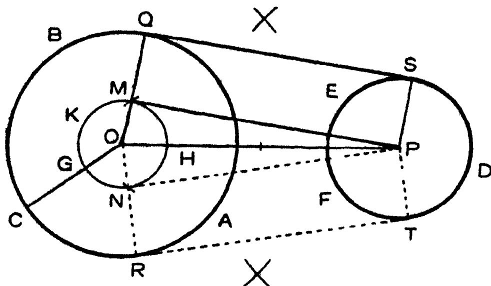
Hence, ∵ SP is at right angles to the radius OP through a point, P, on the circle PQR, SP is a tangent at P to the ⊙PQR.

Obs. Since S is without and O is within the ⊙PQR, the ⊙SPQ, described on SO as diameter, must intersect the ⊙PQR in two different points P and Q. Thus, SQ is also a tangent to the ⊙PQR, because the ∠SQO, being in a semi-⊙, is a rt. angle.

Hence, SP and SQ are the two tangents drawn from S to the ⊙PQR. *Cf. Th. 45A.*

PROBLEM 33.

To draw a common tangent to two given circles.



Let ABC and DEF be two given ⊙ˢ whose centres are O and P respectively, the radius of the first ⊙ being greater than the radius of the second.

It is required to draw a common tangent to the ⊙ˢ ABC *and* DEF.

Construction. Draw any radius OC of the ⊙ABC

(i) From CO cut off CG = the radius of the ⊙DEF.

With O as centre and OG as radius describe the ⊙GKH.

Draw PM, PN tangents to the ⊙GKH, from P

Join OM, ON and produce them to meet the ⊙ABC at Q, R respectively.

In the ⊙DEF, draw the radius PS ∥ to OQ, and the radius PT ∥ to OR.

Join QS, RT.

Then QS as well as RT are common tangents to the ⊙ˢ ABC and DEF.

Proof. Since PM touches the ⊙GKH at M,

∴ the ∠OMP is a rt. ∠;

and ∴ the ∠PMQ is also a rt. ∠.

Now, OQ = OC, and of these the parts OM, OG are equal;

∴ MQ = CG.

Hence PS = MQ; and PS is also ∥ to MQ, by construction;

∴ PSQM is a parallelogram.

Also, since the ∠PMQ of this parallelogram is a rt. ∠.

∴ all its ∠ˢ are rt. ∠ˢ.

Thus the ∠ˢ OQS and PSQ are each a rt. ∠ ;

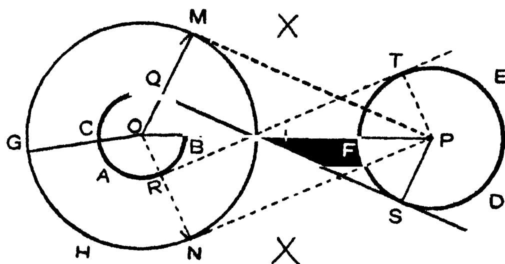
∴ QS is a tangent to the ⊙ABC as well as to the ⊙DEF.

In the same way it may be proved that RT is a tangent to both the given ⊙ˢ.

Q. E. F.

(ii) From OC produced cut off CG = the radius of the ⊙DEF.

With O as centre and OG as radius describe the ⊙GMH.



Draw PM, PN tangents to the ⊙GMH.

Join OM, ON cutting the ⊙ABC at Q and R respectively.

In the ⊙DEF, draw the radius PS ∥ to MO, and the radius PT ∥ to NO.

Join QS, RT.

Then QS as well as RT are common tangents to the ⊙ˢ ABC and DEF.

[The proof is left as an exercise for the student.]

25. Definition : It is clear that there are *four common tangents* to two given ⊙ˢ, if the ⊙ˢ do not cut each other. A common tangent whose pts. of contact are on the same side of the line joining the centres is called a **direct common tangent ;** whilst a common tangent whose pts. of contact are on opposite sides of the line joining the centres is called **a transverse common tangent.** We have *direct common tangents* in the first diagram, and *transverse common tangents* in the second.

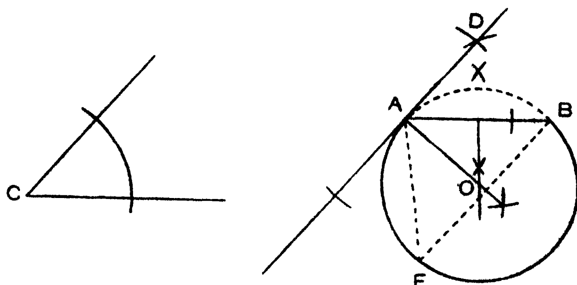
Obs. 1. If the direct common tangents be produced to meet at V, then each of the pts. P and O is equally distant from the lines QV and RV and ∴ both of them lie on the bisector of the ∠QVR.

Hence OP produced passes through V. Thus, *the direct common tangents to two given ⊙s and the line joining their centres are concurrent.*

Obs. 2. In the second diagram, each of the pts. P and O is equally distant from TR and QS, and ∴, if TR and QS intersect at W, both P and O lie on the bisector of the ∠s TWS and QWR. Hence OP passes through W. Thus, *the transverse common tangents to two given ⊙s intersect on the line joining their centres.*

PROBLEM 34.

On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.



Let AB be the given straight line and the ∠C, the given angle.

It is required to draw a segment of a circle containing an angle equal to the ∠C.

Construction. Make the ∠BAD = the ∠C. Let the perpendicular bisector of AB and the perpendicular to AD through A meet at O. Join OA. Since O is on the ⊥ bisector of AB, O is equidistant from A and B.

Hence the segment of the ⊙ described with O as centre and OA as radius will pass through B, and contain an angle equal to the ∠C.

Proof. Now, since AD is ⊥ to the radius OA,

∴ AD touches the ⨀ at A.

Hence the ∠AFB in the segment AFB = the ∠BAD = the ∠C.

Thus, AFB is the required segment.

Q. E. F.

EXERCISE 12

1. Place a chord of length 2″ through a given point on the circumference of a circle of radius 2″.

2. Draw in a circle a chord of given length, making an angle of 45° with another given chord.

3. Describe a circle of radius equal to 6 centimetres and place in it two chords, each of length 7 centimetres. Measure the distances of their middle points from the centre of the circle and verify that they are equal.

4. AB is a chord of length 2″ in a circle whose radius is 2″ and centre is O; through A, AC is drawn at right angles to AB to meet the circle at C. Verify that the points B, O, C lie in one straight line.

5. Describe a circle of radius 13 cm., and from the centre draw a straight line 5 cm. in length. At the other extremity of this line erect a perpendicular chord, and show that its length is 24 cm.

6. Describe two circles each of one inch radius, and place in each a chord one inch long. Verify that the angles subtended by the chords at the respective centres are equal.

7. Describe two circles, each of 2″ radius, and in each draw a pair of radii containing an angle of 120°. Verify that the chords subtending these angles are equal, and that their distances from the respective centres are equal.

8. In a circle of 3″ radius place two chords, each measuring 2″, and verify by actual measurement with the help of a protractor, that the angles which they subtend at the centre, are equal.

9. A pair of tangents is drawn to a circle of 3″ radius from a point distant 5″ from the centre. Show by actual measurement that

their lengths as also the length of the chord of contact are 4″ and 4⅛″ respectively.

10. Draw a tangent to a circle, of diameter 6″, from a given point, distant 4″ from the centre; how many solutions are there?

11. Draw a tangent to a circle of 2″·5 radius from a point, distant 6″·5 from the centre, and verify that its length is equal to 6″; also that the length of the chord of contact is 4″·6 nearly.

12. Describe two circles of radii equal to 2″ and 3″ respectively, touching each other externally, at a given point.

13. Describe three equal circles of radius 1″ touching one another externally.

14. Describe three circles of given radii touching one another externally.

15. The centres of two circles are 6½ inches apart, and radii are 3½″ and 1″ respectively. Draw their direct common tangents, and show that their lengths are each equal to 6″.

16. Describe two circles of 2½″ and 1½″ radius respectively, having their centres 5″ apart. Draw their direct common tangents. Verify that they meet, when produced, on the line of centres.

17. In the circles of ex. 16, draw the transverse common tangents and show that they intersect on the line of their centres.

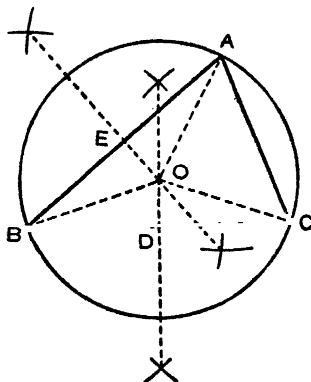
18. The centres of two circles are 5″ apart, and their radii are 3½″ and 1½″ respectively. Draw their direct common tangents. Are there *two* transverse common tongents?

19. On a straight line, 2″ long, describe a segment of a circle containing an angle of (i) 45°, (ii) 60°.

20. Construct a circle circumscribing a triangle one of whose angles is equal to a given angle.

PROBLEM 35.

To circumscribe a circle about a given triangle.



Let ABC be the given triangle.

It is required to draw a circle about the △ ABC.

Construction. Bisect the sides AB and BC at rt. angles by OE and OD respectively and let them meet at O. Join OB. With centre O and radius OB draw a circle.

This circle will pass through A, B, C and is, therefore, the required one.

Proof. Because O is a point on the perp. bisector of BC,

∴ OB = OC.

Similarly, OB = OA.

∴ OA = OB = OC.

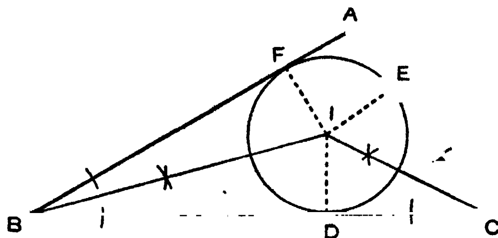
∴ the ⊙ with centre O and radius OB will pass through A and C, and is, therefore, the required circum-circle. Q. E. F.

NOTE 1. The circum-centre of a triangle lies *within* or *without* the triangle according as it is *acute-angled* or *obtuse-angled*. When the triangle is a *right-angled* one, the circum-centre is the middle point of the hypotenuse.

NOTE 2. It may be easily seen that if O be joined to the middle point F of AC, OF is perp. to AC. Thus, *the three perp. bisectors of the sides of a triangle meet in a point, which is the circum-centre of the triangle.*

PROBLEM 36.

To inscribe a circle in a given triangle.



Let ABC be the given triangle.

It is required to draw a circle which will be touched by all the sides ***internally.***

Construction. Bisect the ∠ˢ ABC, ACB by BI and CI, which meet at I.

From I, draw ID ⊥ to BC.

Then, the ⊙ with centre I and the radius ID is the required one.

Proof. From I draw IE and IF perpˢ on CA and AB respectively.

Because I is on the bisector BI, of the ∠ ABC,

ID = IF.

Similarly, ID = IE, [∵ I lies on CI.]

∴ ID = IE = IF.

∴ the ⊙ drawn with the centre I and the radius ID passes through E and F.

Again, because BC is perp. to the radius ID, BC touches the circle at D.

Similarly, CA and AB touch the ⊙ at E and F respectively.

∴ the ⊙ DEF is inscribed in the Δ ABC.

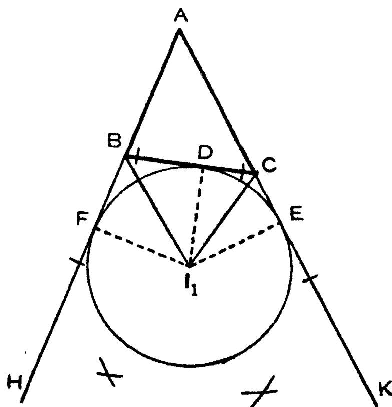
Obs. It may be easily shown by joining A to I that AI bisects the ∠BAC. Thus, *the three bisectors of the angles of a triangle meet in a point, which is the centre of the circle inscribed in the triangle.*

26. Definitions : (1) The circle inscribed in a triangle is called the **in-circle,** its centre, the **in-centre** and its radius, the **in-radius,** of the triangle.

(2) The circle which is touched by *only one* side of a triangle and two other *sides produced* is called an **escribed circle,** shortly an **ex-circle,** of the triangle; and its centre and radius are respectively called an **ex-centre** and an **ex-radius** of the triangle.

PROBLEM 37.

To draw an escribed circle of a given triangle.



Let ABC be the given triangle, of which the sides AB and AC are produced to H and K respectively.

It is required to draw the circle touching BC, *and* AB, AC *produced.*

Construction. Bisect the $\angle$ˢ HBC, KCB by BI_1 and CI_1 respectively, which intersect at I_1. Draw I_1D, I_1E, I_1F perp. on BC, AK, AH respectively. The circle drawn with the centre I_1 and radius I_1F is the required one.

Proof. Because I_1 is a point on BI_1, $I_1D = I_1F$; similarly, $I_1D = I_1E$, $\because$ I_1 lies on CI_1.

$\therefore$ $I_1D = I_1E = I_1F$. Hence, the ⊙ with I_1 as centre and I_1F as radius passes through D and E.

Again, since BC, CK, BH are perpˢ to the radii I_1D, I_1E, I_1F, the ⊙ is touched by BC, CK, BH. **Q. E. F.**

Obs. 1. It may be shown that AI_1 bisects the $\angle BAC$; thus, *the internal bisector of one angle of a triangle and the external bisectors of the other two, meet in a point, which is an ex-centre of the triangle.*

Obs. 2. By producing the sides BC, BA, another ex-circle may be drawn touching the side CA and BC, BA, produced. Similarly, there is another ex-circle touching AB and CB, CA produced. Thus, *there are three ex-circles of a triangle each one touching one of its sides and the two others produced.*

EXERCISE 13

1. Draw a circle to touch each of two given parallel straight lines and a transversal intersecting them.

2. Two straight lines are *almost* parallel. Bisect the angle between them, giving all lines of construction.

3. If I be the in-centre of a triangle ABC, prove that

$$\angle BIC = 90 + \frac{A}{2}.$$

4. If I_1, be the ex-centre of the triangle ABC opposite to the angle A, prove that

$$\angle BI_1C = 90 - \frac{A}{2}.$$

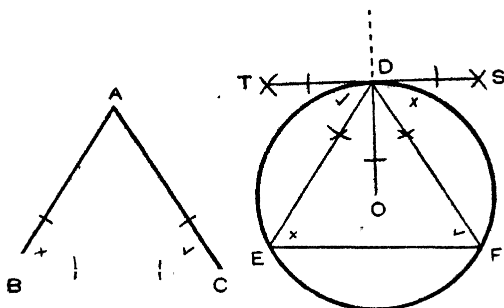
5. The inscribed circle of the triangle ABC touches BC at D, and the circle escribed to the triangle, opposite to A, touches BC at D_1. If $AB > AC$ and X is the middle point of BC, prove that

$$XD = XD_1 = \tfrac{1}{2}(AB - AC).$$

6. Show that the bisectors of the vertical angles of triangles standing on a given base and having the sum of their base angles equal to a given angle are concurrent.

PROBLEM 38.

In a given circle to inscribe a triangle equiangular to a given triangle.



Let be O the centre of the given ⊙, and ABC the given △.

It is required to inscribe in the circle (O) *a triangle equiangular to the triangle* ABC.

Construction. Take any radius OD, and through D draw TS ⊥ to OD; then TS is the tangent at D.

Draw the chord DE making the ∠TDE = the ∠ACB; and draw the chord DF making the ∠SDF = the ∠ABC.

Join EF.

Then DEF is the reqd. △.

Proof. Since TS touches the ⊙ at D,

∴ the ∠TDE = the ∠DFE, in the alt. segment;

and the ∠SDF = the ∠DEF, in the alt. segment.

Hence the ∠DFE = the ∠ACB.
and the ∠DEF = the ∠ABC;

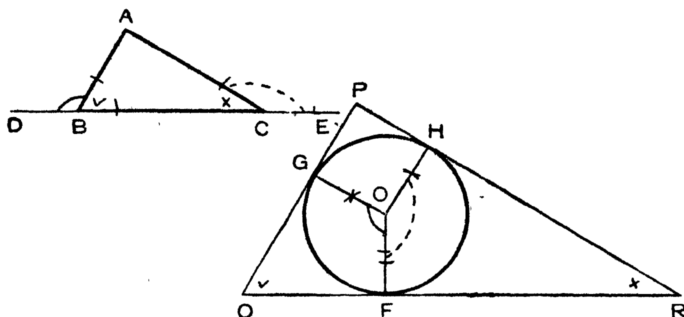
∴ the remaining ∠EDF = the remaining ∠BAC.

Thus, the △DEF inscribed in the given ⊙ is equiangular to the △ABC. Q. E. F.

NOTE. To inscribe an equilateral triangle in a given circle, all that we have to do is to construct an equilateral triangle outside the circle and then to inscribe in the circle a triangle equiangular to the triangle constructed, in the manner indicated above. In this case therefore, each of the angles TDE and SDF would be equal to an angle of an equilateral triangle, *i.e.* equal to one-third of two right angles.

PROBLEM 39.

About a given circle to circumscribe a triangle equiangular to a given triangle.



Let O be the centre of the given ⊙, and ABC the given △.

It is required to circumscribe about the circle (O) *a triangle equiangular to the triangle* ABC.

Construction. Produce BC both ways to D and E.
Take any radius OF of the ⊙.

Draw the radius OG making the ∠ FOG = the ext. ∠ ABD; and draw the radius OH making the ∠ FOH = the ext. ∠ ACE.

Draw tangents to the ⊙ at the pts. F, G, H, forming the △ PQR.

Then PQR is the reqd. △.

Proof. The ∠s OFQ and OGQ being rt. ∠s, the remaining ∠s FOG and GQF of the quadl. OFQG are supplementary.

Now, the ∠ ABC is the supplement of the ∠ ABD, and the ∠ GQF is the supplement of the ∠ FOG;

but the ∠ ABD = the ∠ FOG;
∴ the ∠ ABC = the ∠ PQR.
Similarly, the ∠ ACB = the ∠ PRQ.

∴ the remaining ∠ QPR = the remaining ∠ BAC.

Thus, the △ PQR circumscribed about the given ⊙ is equiangular to the △ ABC. Q. E. F.

NOTE 1. To circumscribe an equilateral triangle about a given circle, all that we have to do is to construct an equilateral triangle outside the circle and then circumscribe about the circle a triangle equiangular to the triangle constructed, in the manner indicated above. In this case, therefore, each of the angles FOG, FOH would be equal to the supplement of an angle of an equilateral triangle, *i.e.*, equal to two-thirds of two right angles.

NOTE 2. An equilateral triangle is evidently a *regular* figure of three sides, for an equilateral $\triangle$ is also equiangular. Hence, we know how to construct a regular figure of three sides in or about a given $\odot$.

EXERCISE 14

(*On Circles in connection with triangles.*)

1. Draw a circle of 1″ radius and inscribe an equilateral triangle in it. Show that the length of a side of the triangle is 1·73 inches.

2. Circumscribe a circle about a triangle whose sides are 3″, 4″ and 5″. Measure the length of its radius.

3. Explain how to construct a triangle, having given the length of its circum-radius and two of its angles.

4. ABC is an equilateral triangle, whose sides are each 2″ in length. Draw the escribed circle touching the side CA. Measure the length of its radius.

5. Draw a circle of radius 1″ and describe an equilateral triangle about it. Show that the length of a side of the triangle is 3·46″.

6. Compare the areas of the triangles in Ex. 1 and Ex. 5.

7. Draw a circle of 1″ radius. Inscribe in it, and describe about it, triangles equiangular to a triangle whose sides are 1·3″, 1·4″ and 1·5″. Show that the greatest sides of the triangles so drawn, are 1·84″ and 3·75″, respectively.

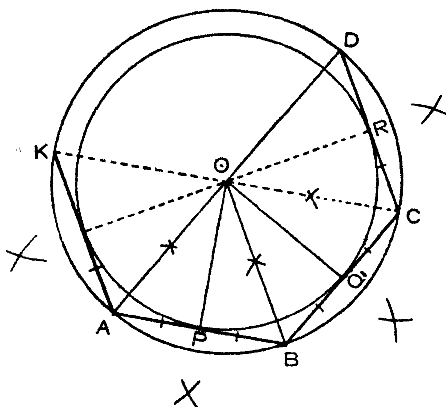
8. If I, I_1, I_2, I_3 be the in-centre and the three ex-centres opposite respectively to the angles A, B, C of a triangle ABC, and the sides opposite to these angles in order be a, b, c, verify that I_2, A, I_3 ; I_3, B, I_1 ; I_1, C, I_2 are collinear.

ON CONSTRUCTION OF CIRCLES IN CONNECTION WITH REGULAR POLYGONS

Definition. A *polygon* is said to be **regular** when all its sides and angles are equal.

PROBLEM 40.

To draw a circle (i) ***in,*** *or (ii)* ***about,*** *a regulur polygon.*



Let ABCD......K be a regular polygon of n sides.

(i) It is required to inscribe a circle about it ;
and (ii) to circumscribe a circle about it.

Construction. Bisect the ∠ˢ A, B by AO, BO which meet at O, suppose.

Join OC, OD,...OK.

Also, from O, draw OP, OQ, OR,......perpˢ on AB, BC, CD,......respectively.

Then, (i) the circle with O as centre and OP as radius is the required inscribed circle ;

and (ii) the circle with the centre O and radius OA is the required circumscribed circle.

Proof. Because, in the Δˢ OBA, OBC,

OB, BA = OB, BC and the ∠OBA = the ∠OBC,

∴ OA = OC ;

and the ∠OCB = the ∠OAB = $\frac{1}{2}$ the ∠KAB = $\frac{1}{2}$ the ∠BCD.

∴ OC bisects the ∠BCD.

Similarly, OD bisects the ∠D, and so on.

Thus, the bisectors of all the angles meet at O.

Hence, $\because$ the $\angle A$ = the $\angle B$ = the $\angle C$,..,
we have $OA = OB = OC = ...$

Also, since O lies on the bisectors of the $\angle^s$ A, B, C,... it is equidistant from the sides.

$\therefore$ $OP = OQ = OR = ...$

$\therefore$ (i) the circle with O as centre and OP as radius passes through Q, R,......and touches the sides AB, BC,...... because AB, BC,...are perp^s to the radii OP, OQ,...

and (ii) the circle with O as centre and OA as radius passes through B, C,....

Q. E. F.

EXERCISE 15

1. The base of an isosceles triangle is 14″ and each side is 25″ show that the length of the in-radius is 5·25″.

2. Prove that the in-radius of an equilateral triangle is equal in length to one-third of its height.

3. Inscribe a circle in a given square and verify that the diameter of the circle is equal to a side of the square.

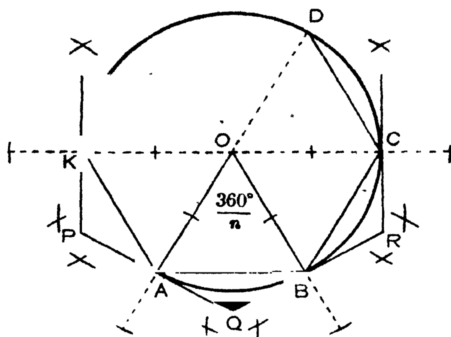
4. Inscribe a circle in a given rhombus and show that the diameter of the circle is equal to the height of the rhombus.

5. Show that the circum-radius of an equilateral triangle is two-thirds of its height.

6. Construct the circum-circle of a triangle whose sides are 7, 8, 9 yds. Verify that the length of the radius of the circle is 4 yds. 2 ft. 1·04 in.

PROBLEM 41.

To draw a regular polygon (i) ***in,*** *or (ii)* ***about,*** *a given circle.*



Let ABC be a given circle whose centre is O.

(i) To inscribe a regular polygon of n sides in the circle, and (ii) to circumscribe a regular polygon of n sides about the circle.

(i) **Analysis.** Suppose ABCD...to be a regular polygon inscribed in the ⊙, so that the sides AB, BC, CD,......are the consecutive chords of the circle.

Hence, since AB = BC = CD = ...,

$$\angle AOB = \angle BOC = \angle COD = \ldots.$$

Thus, the angle at O is divided into as many equal parts as the polygon has sides.

$\therefore \angle AOB = \angle BOC = \angle COD = \ldots = \frac{360°}{n}$, *n being the no. of sides.*

Hence the following construction :—

Construction. Join O to any point A, on the $\bigcirc^{ce}$; at O, draw a radius OB, making the $\angle AOB = \frac{360°}{n}$; join AB.

Place consecutive chords BC, CD,..., each equal to AB, round the whole $\bigcirc^{ce}$.

The figure, so formed, is the required *regular* polygon.

(ii) Join O to any pt. A on the $\bigcirc^{ce}$.

At O, draw radii OB, OC, OD,...round the whole $\bigcirc^{ce}$, making the $\angle$ AOB = the $\angle$ BOC = the $\angle$ COD = ... = $\frac{360^\circ}{n}$.

At A, B, C, D...draw tangents to the ⊙ meeting each other at P, Q, R, S,...

Then PQRS...is the reqd. regular polygon circumscribing the ⊙.

[The proof is obvious and is left as an exercise to the student.]

Obs. It is to be observed that such constructions are not *possible* purely with the help of a *ruler* and a *compass* except for certain *special* values of *n*. As for example, as there is no method of constructing an angle equal to $\frac{1}{7}$ of 360°, purely with *ruler* and *compass*, a regular *heptagon* (*i.e.* of seven sides) can neither be inscribed in or circumscribed about a given circle, except with the use of a protractor.

EXERCISE 15A

1. *Construct a square (i) in or (ii) about a given circle.*

[Let O be the centre of the ⊙. Take any diameter and draw another diameter ⊥ to the former, The figure obtained by joining the extremities is a square inscribed in the circle. Again, the figure formed by the tangents at the extremities is a circumscribing square].

2. *Construct a regular hexagon in or about a given circle.*

[$\angle$AOB in this case = 60° ; hence, place consecutive chords AB, BC, CD,...each equal to the radius. The fig. ABCD...is the reqd. inscribed hexagon. Draw tangents at A, B, C, D...to construct the circumscribing hexagon.]

3. Construct a regular polygon (*i.e.* of 8 sides) in or about a given circle.

4. Construct a regular dodecagon (*i.e.* of 12 sides) in or about a given circle.

5. Construct a regular polygon of (i) 16, (ii) 24, sides in or about a given circle.

6. Show by actual measurement that the perimeter of a regular hexagon inscribed in a circle of radius 4″ is 2 ft.

7. About a circle, 2″ in diameter, describe a regular hexagon ; verify, by actual measurement and calculation, that the area of the hexagon is 3·46 sq. inches.

*8. Compare the areas of inscribed and circumscribed regular hexagons of a given circle.

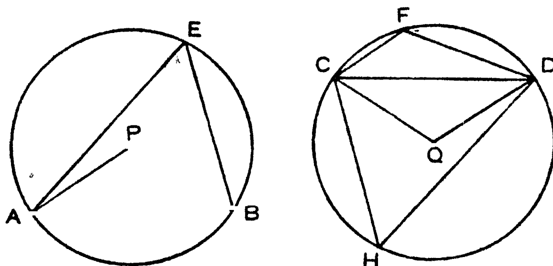
9. Each side of an octagonal field is 100 yds. ; show that the price of the field at Rs. 225 per acre is Rs. 2244. 9*a*. 9*p*.

CHAPTER VI

MISCELLANEOUS PROPOSITION

I. On Circles and Triangles.

If two triangles standing on equal bases have vertical angles that are either equal or supplementary, then their circum-circles are equal.



Let the Δ^s AEB, CFD standing on equal bases AB, CD have their vertical $\angle^s$ AEB, CFD *supplementary.*

Let P, Q be the centres of the $\odot^s$ circumscribed about the Δ^s AEB, CFD respectively.

It is required to prove that these two circles have equal radii.

Proof. Join PA, PB, QC, QD.

Take any pt. H on the arc conjugate to the arc CFD. Join CH, DH.

Then the $\angle$CHD = the $\angle$AEB,

each of them being supplementary to the $\angle$CFD ;

$\therefore$ the $\angle$CQD = the $\angle$APB, (*double of equal* $\angle^s$).

Now, since the $\angle$PAB = the $\angle$PBA, each of them = $\frac{1}{2}$ their sum = $\frac{1}{2}$ the supplement of the $\angle$APB.

Similarly, each of the $\angle^s$ QCD, QDC

= $\frac{1}{2}$ the supplement of the $\angle$CQD.

Hence, each of the Δ^s PAB, PBA = each of the $\angle^s$ QCD, QDC.

Thus, the two isos. Δ^s PAB, QCD, have their bases equal, and also the $\angle^s$ at the base of the one = the $\angle^s$ at the base of the other ;

$\therefore$ these two Δ^s are congruent.

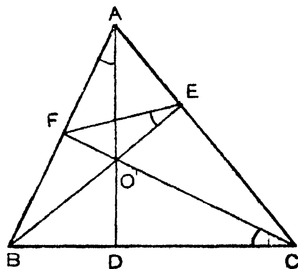
Hence, each of the sides PA, PB = each of the sides QC, QD.

Thus, the radii of the two $\odot^s$ are equal, which proves the proposition. Q. E. D.

NOTE. It is obvious, from the foregoing demonstration, how to deal with the case when the two triangles have their vertical angles *equal*.

II. Ortho-centre : Pedal Triangle.

1. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*



Let BE, CF be $\perp^s$ to the sides, CA, AB respectively, of the ΔABC ; and let them intersect at O.

Join AO and produce it to meet BC at D.

It is required to prove that AD *is* $\perp$ *to* BC.

Proof. Join FE.

The $\angle^s$ OFA, OEA are supplementary, each of them being a rt. $\angle$;

$\therefore$ the quadl. AFOE is cyclic,

$\therefore$ the $\angle$FAO = the $\angle$FEO, in the same segment.... (α)

Again, the ∠BFC = the ∠BEC, each being a rt. ∠ ;

∴ the four pts. B, F, E, C are concyclic,

∴ the ∠FEB = the ∠FCB, in the same segment.... (β)

Hence, from (α) and (β),

the ∠BAD = the ∠BCF.

Now in the Δˢ BAD, BCF, the ∠ at B is common, and the ∠BAD = the ∠BCF ;

∴ the remaining ∠ADB = the remaining ∠CFB = a rt. ∠.

Hence, AD is ⊥ to BC.

Q. E. D.

NOTE. For an alternative proof, see Book I, p. 110.

27. Definitions. (i) The point of intersection of the perpendiculars from the vertices of a triangle to the opposite sides is called the **ortho-centre** of the triangle.

Thus, in the preceding diagram, the point O is the *ortho-centre* of the triangle ABC.

(ii) The triangle formed by joining the feet of the perpendiculars drawn from the vertices of any given triangle to the opposite sides is called the **pedal triangle** of the given triangle.

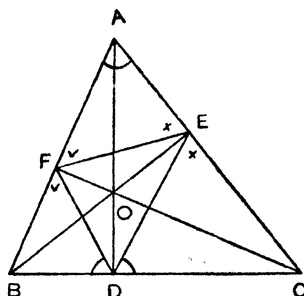
In the preceding diagram, the triangle DEF is the *pedal triangle* of the triangle ABC.

COR. *If* O *is the ortho-centre of a triangle* ABC, *the circum-circles of the four triangles* ABC, BOC, COA, AOB *are equal to one another.*

In the preceding diagram the quadl. AFOE being cyclic, the ∠FAE is supplementary to the ∠FOE and is ∴ supplementary to the ∠BOC. Thus, the ∠ˢ BAC, BOC stand on the same base and have their vertical ∠ˢ supplementary ;

∴ the circum-⊙ of the ΔBOC is = that of the ΔABC. Similarly, the circum-⊙ of each of the Δˢ COA, AOB is = that of the ΔABC. Hence the four circum-⊙ˢ are equal to one another.

2. *In an acute-angled traingle the perpendiculars drawn from the vertices to the opposite sides bisect the angles of the pedal triangle.*



Let O be the ortho-centre, and DEF the pedal Δ, of the acute-angled Δ ABC.

It is required to prove that AD, BE, CF *bisect the* ∠ˢ D, E, F *of the* Δ DEF, *respectively.*

Proof. The ∠ ADB = the ∠ AEB, each being a rt.∠.

∴ the quadl. AEDB is cyclic.

Hence, the ∠EDC = the ∠BAE ;
and similarly, the ∠FDB = the ∠CAF,
∴ the ∠EDC = the ∠FDB ;
and the whole ∠ ADC = the whole ∠ ADB, each = 90°.
∴ the remaining ∠ ADE = the remaining ∠ ADF.

Thus, AD bisects the ∠EDF.

Similarly, BE and CF bisect the ∠ˢ DEF and DFE respectively. Q. E. D.

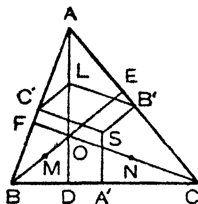
COR. 1. *The sides of the pedal triangle are equally inclined to that side of the original triangle on which they meet.*

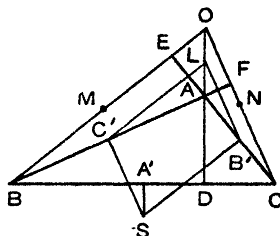
[*See proof of the Theorem*]

COR. 2. *The triangles* AFE, BDF, CDE *are equiangular to one another and to the original triangle.*

[*See proof of the Theorem.*]

3. *The distance of each vertex of a triangle from the ortho-centre is double the distance of the circum-centre from the opposite side.*





Let O be the ortho-centre of the $\triangle$ ABC ; AD, BE, CF being the $\perp^s$ from A, B, C on the opposite sides.

Let S be the circum-centre of the $\triangle$ ABC, and A′, B′, C′ the mid-pts. of the sides opposite to A, B, C respectively.

Then SA′, SB′, SC′ are $\perp^s$ respectively to BC, CA, AB.

It is required to prove that AO = 2SA′, BO = 2SB′, CO = 2SC′.

Proof. SA′ is ∥ to AD, because both of them are $\perp^s$ to BC.

Similarly, SB′ is ∥ to BE, and SC′ is ∥ to CF. Let L, M, N be the mid-pts. of AO, BO, CO respectively.

Join LB′, LC′.

Now, B′L passes through the mid-pts. of CA and OA ;

∴ B′L is ∥ to CO and ∴ to SC′ ;
similarly, C′L is ∥ to BO and ∴ to SB′.

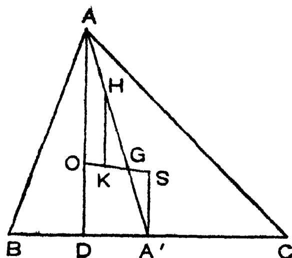
Hence the quadl. SL is a par^m ;

∴ $SB' = C'L = \frac{1}{2}BO,$
and $SC' = B'L = \frac{1}{2}CO.$

Similarly, by joining MC′, MA′, (*or* NA′, NB′), it can be shewn that $SA' = \frac{1}{2}AO$. Q. E. D.

NOTE. If O is the ortho-centre of any given $\triangle$ABC, the following constructions for the circum-centre shou be carefully noted. Through A′, the mid-pt. of the side opp. to A, draw A′S ∥ to, and in the same *sense* as, OA ; make $A'S = \frac{1}{2}OA$. Then S is the circum-centre of the given $\triangle$.

COR. *The circum-centre, the centroid and the ortho-centre of a triangle are collinear.*



Let S be the circum-centre, and O the ortho-centre, of the △ABC ; and let A′ be the mid-pt. of the side BC Join SA′ and SO ; also join AA′, cutting OS in G. Let H, K be the mid-pts. of AG and OG respectively ; join HK. Now HK is ∥ to AO and $=\frac{1}{2}$AO ; ∴ HK is equal and ∥ to SA′, and ∴ the quadl. HKA′S is a par^m. But the diagonals of a par^m bisect each ether ; ∴ A′G = GH $=\frac{1}{2}$AG. Hence, G is the centroid of the △ABC, and it lies on the line SO, which proves the corollary.

EXERCISE 16

1. If O be the ortho-centre of a triangle ABC and if AO be produced to meet the circum-circle at P, show that OP is bisected at right angles by BC.

2. If O be the in-centre of a triangle ABC, and if AO, BO, CO be produced to meet the circum-circle in P, Q, R respectively, then O is the ortho-centre of the triangle PQR.

[∠PRC=∠PAC=½∠A ; similarly, ∠CRQ=½∠B and ∠RPA=½∠C ; hence, if QR meet AO in D, ∠PRD+∠RPD=½∠A+½∠B+½∠C=one rt. ∠ ; ∴ ∠PDR=one rt. ∠ ; hence etc.]

3. *If* O *be the ortho-centre of a triangle* ABC, *show that* A, B, C *are the ortho-centres of the triangles* BOC, COA *and* AOB *respectively.*

4. *Deduce from Ex. 3 that the circum-circles of the triangles* BOC, COA, AOB *are equal, each being equal to that of the triangle* ABC.

[Let S be the circum-centre and A′, the mid-pt. of BC. Then, SA′ par^l to AO and=½AO. Produce SA′ to P, making A′P=SA′ (=½AO); then P is the circum-centre of the △BOC. Also, POAS being evidently a par^m, PO=SA=circum-radius of △ABC; hence etc.]

5. If O is the ortho-centre and S the circum-centre of the triangle ABC and if A′, B′, C′ are the circum-centres of the triangles OBC, OCA, OAB, show that O is the circum-centre and S the ortho-centre of the triangle A′B′C′. [Apply *Ex* 3, 4.]

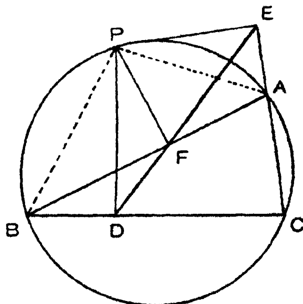
6. Show that the angles of the pedal triangle of the triangle ABC are supplements of the angles 2A, 2B, 2C.

7. Construct a triangle having given a vertex, the circum-centre and the ortho-centre.

8. Construct a triangle having given the circum-centre, the ortho-centre and the middle point of the base.

III. Simson's Line or the Pedal Line.

If perpendiculars be drawn to the sides of a triangle from any point on the circum-circle, then the feet of these perpendiculars lie in one straight line.



From any pt. P on the circum-⊙ of the △ABC, let PD, PE, PF be drawn ⊥^{s} to BC, CA, AB respectively.

Join DF, FE.

It is required to prove that DF, FE *are in one str. line.*

Proof. Join PA, PB.

Since the ∠PDB = the ∠PFB, each being a rt. ∠, the quadl. PFDB is cyclic.

Also, since the $\angle^s$ PEA and PFA are supplementary, each being a rt. $\angle$, the quadl. PFAE is cyclic.

Now, the quadl. PBCA being cyclic, the $\angle$PBC = the $\angle$PAE ;

and, the quadl. PFAE being cyclic, the $\angle$PAE = the $\angle$PFE.

Hence, the $\angle$PBD = the $\angle$PFE.

But, the quadl. PFDB being cyclic, the $\angle^s$ PBD and PFD are supplementary :

$\therefore$ the $\angle^s$ PFE and PFD are supplementary; $\therefore$ DF, FE are in one str. line.

Q. E. D.

28. Definition. The line DFE is called **pedal line** of the point P with respect to the triangle ABC.

EXERCISE 17

1. If the feet of the perpendiculars from a point to the sides of a triangle be collinear, prove that the point lies on the circum-circle of the triangle.

2. From a point P on the circum-circle of a triangle ABC, PL and PM are drawn perps. to the sides BC and CA; if LM, produced, meets AB (produced if necessary) at N, show that PN is perp. to AB.

3. From a point P on the circum-circle of a triangle ABC, PL is drawn perp. to the side BC and is produced to meet the circum-circle at Q ; show that AQ is parallel to the *pedal line* of P.

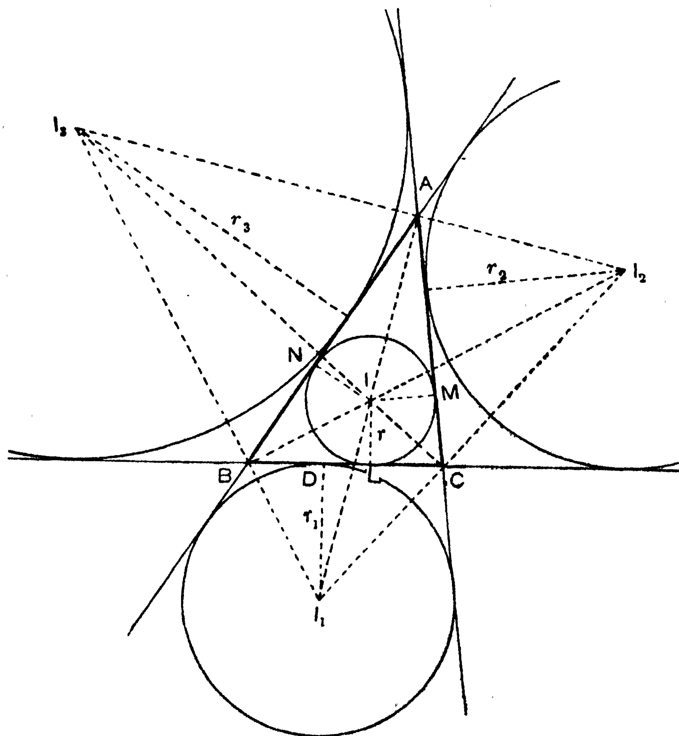
4. If, in a triangle ABC, the *pedal line* of a point P meets the side AC, the angle which the *pedal line* makes with AC is complementary to what the arc BP subtends at any point on the circumference.

5. The angle between the *pedal lines* of any two points P and Q on the circum-circle of a triangle is equal to the angle subtended by the arc PQ at the circumference.

EXERCISE 18

(On the Inscribed and Escribed Circles of a Triangle.)

1. If I be the in-centre and I_1, I_2, I_3 be the three ex-centres of a triangle ABC,



prove that

(*i*) A, I, I_1 are collinear ;
(*ii*) B, I, I_2 ,, ,,
(*iii*) C, I, I_3 ,, ,,

2. Prove that

(*i*) I_3, A, I_2 lie on a str. line;
(*ii*) I_2, C, I_1 „ „ „ „ „ ;
(*iii*) I_1, B, I_3 „ „ „ „ „ .

3. Shew that I_1A, I_2B, I_3C are at right angles to I_2I_3, I_3I_1, I_1I_2 respectively.

4. Show that the triangle ABC is the pedal triangle of the $\Delta I_1I_2I_3$.

5. Show that the angles of the triangle $I_1I_2I_3$ are

$$90° - \frac{A}{2},\ 90° - \frac{B}{2},\ 90° - \frac{C}{2}.$$

6. If the in-circle of the ΔABC touch the sides BC, CA, AB at L, M, N respectively, and if $2s$ represent the perimeter of the triangle, prove that

(*i*) $AN = AM = s - a$;
(*ii*) $BN = BL = s - b$;
(*iii*) $CL = CM = s - c$, where a, b, c denote the lengths of the sides BC, CA, AB respectively.

7. If the escribed circle opposite to the vertex A touch BC at D and AB and AC produced at F and E respectively, show that AF= AE=semi-perimeter of the triangle.

8. If S represent the area of the triangle ABC, and r, r_1, r_2, r_3 denote the in and ex-radii of the triangle in order, prove that

$$(i)\ r = \frac{S}{s};\ (ii)\ r_1 = \frac{S}{s-a};\ (iii)\ r_2 = \frac{S}{s-b};\ (iv)\ r_3 = \frac{S}{s-c}.$$

Prove that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}.$$

9. *Show that the ortho-centre of a triangle is the in-centre of its pedal triangle.*

10. Given the base and the magnitude of the vertical angle, find the locus of the ex-centre opposite to the vertex.

11. In the diagram of Ex. 1, show that II_1, II_2, II_3 are each bisected by the circum-circle of the triangle.

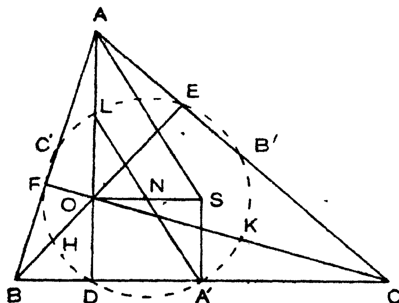
12. Construct a triangle of which the three ex-centres are given in position.

13. Given the vertical angle, perimeter and radius of the inscribed circle; construct the triangle.

14. Given the vertical angle, the in-radius, and the length of the perpendicular from the vertex to the base, construct the triangle.

IV. Nine-point Circle.

In any triangle, the middle points of the sides, the feet of the perpendiculars from the vertices to the opposite sides, and the middle points of the lines joining the ortho-centre to the vertices are concyclic.



In the above diagram, O is the ortho-centre and S the circum-centre of the △ABC; A′, B′, C′ are the mid-pts. of the sides opp. to A, B, C; AD, BE, CF are the ⊥ˢ from A, B, C to the opp. sides; and L, H, K are the mid-pts. of the lines OA, OB, OC.

It is required to prove that the nine points A′, B′, C′, D, E, F, L, H, K *are concyclic.*

Proof. Join SA, SA′, SO, and LA′ cutting SO at N.

Now, SA′ is ∥ to AO and $=\frac{1}{2}$AO;

∴ SA′ is equal and ∥ to LO.

Hence L, O, A′, S are the consecutive angular pts. of a parallelogram.

Hence LA′ and OS bisect each other at N.

Thus LA′ is bisected at the mid-pt. of OS; and similarly, HB′ and KC′ are each bisected at the mid-pt. of OS.

Now, LDA′ being a rt. ∠, the ⊙ described on LA′ diameter passes through D.

∴ ND = NL = NA′, each of them being a radius of this ⊙.

Again, since SA′ is equal and ∥ to AL.

∴ SA = A′L.

Hence, if R stands for "the circum-radius" of the △ ABC,

we have ND = NL = NA′ = $\frac{1}{2}$R. (SA being R.)

Similarly, NE = NH = NB′ = $\frac{1}{2}$R,
and NF = NK = NC′ = $\frac{1}{2}$R.

Thus, the nine pts. D, L, A′, E, H, B′, F, K, C′ are each at a distance = $\frac{1}{2}$R from N, the mid-pt. of OS;

and ∴ they all lie on the O^{ce} of the ⊙ whose centre is the mid-pt. of OS and radius = $\frac{1}{2}$R. Q. E. D.

29. Definition. The circle passing through the mid-points of the sides of a triangle is called the **nine-point circle** of the triangle, from its property of passing through *nine* particular points connected with the triangle. The centre of the nine-point circle may also be conveniently called the **"nine-point centre."**

Obs. It may be easily seen from the above theorem that

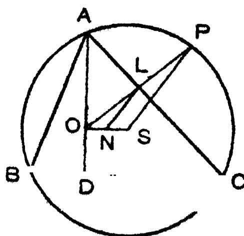
(i) the *nine-point centre* is the middle point of the line joining the circum-centre to the ortho-centre;

(ii) the *radius* of the *nine-point circle* is equal to half the circum-radius.

COR. 1. *The nine-point circle of each of the triangles* BOC, COA, AOB *is the same as that of the triangle* ABC.

COR. 2. *If* P *is any point on the circum-circle of the triangle* ABC, *the mid-point of* OP *is on the nine-point circle.*

If L is the mid-pt. of OP, join NL. Then NL is ∥ to SP, and = $\frac{1}{2}$SP = $\frac{1}{2}$ R. Thus NL is = a radius of the nine-pt. ⊙, and ∴ the pt. L lies on that ⊙.



COR. 3. *Given the base and the vertical angle of a triangle; as the triangle changes its position the centre of the nine-point circle moves along the circumference of a circle whose centre is the mid-point of the given base.*

In the figure of the proposition, suppose XBC to be *any other* position of the triangle, the base BC remaining fixed and the vertical ∠BXC being = the ∠BAC. Then X must be on the arc BAC of the circum-⊙ of the △ABC. Now, if N′ be the nine-pt. centre of the △XBC, we must have $N'A' = \frac{1}{2}SX = \frac{1}{2}SA$. Thus, N and N′ are both at the same distance from A′, which proves the corollary.

EXERCISE 19

1. Show that the circum-centre, the centroid, the ortho-centre and the nine-point centre are collinear.

2. If I, I_1, I_2, I_3 be the centres of the inscribed and escribed circles of the triangle ABC, prove that the circum-circle of the triangle ABC is the nine-point circle of each of the triangles

$$II_1I_2,\ II_2I_3,\ II_3I_1,\ I_1I_2I_3.$$

3. If O be the ortho-centre of the triangle ABC, in which AB > AC and L, U, M be the middle points of BC, AO, AC, prove that

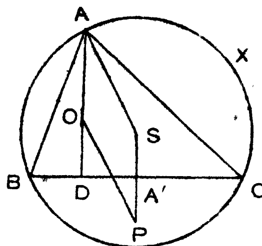
$$\angle LUD = \angle LMD = C - B,$$

4. Lines are drawn through the middle points of the sides of a triangle ABC perpendicular to the bisectors of the opposite angles. Prove that the triangle A′B′C′ formed by these lines has the same nine-point circle as the original triangle.

[If L, M, N be the mid-pts, of BC, CA, AB, prove that LMN is the pedal triangle of A′B′C′].

V. Locii.

1. *Given the base and the vertical angle of a triangle, prove that the locus of its ortho-centre is a circle of which the centre is a known point on the perpendicular bisector of the given base.*



Let BC be the given base of a triangle ABC having its vertical ∠A of given magnitude.

It is required to find the locus of the ortho-centre, O.

Proof. Let AD be the ⊥ on BC, O the ortho-centre, S the circum-centre, and A′ the mid-pt. of BC.

Join SA′ and produce it to P making A′P = SA′.

Join SA, PO.

Now, SA′ is ∥ to AO and $= \frac{1}{2}$AO ;

∴ SP is equal and ∥ to AO.

Hence, PO = SA.

Now, the base BC and the vertical ∠BAC being given, the circum-centre and the circum-radius are the same for all triangles described on BC and having the vertical angle equal to the ∠BAC.

∴ S is a fixed point, and therefore, also, P, for BC is fixed.

Also, PO = SA = the circum-radius, which is also of constant length.

Hence, the locus of O is a circle whose centre is the fixed point, P on the perp. bisector of BC and whose radius is equal to the constant circum-radius. Q. E. D.

Second Method.

Join BO, CO, and produce them to meet the opp. sides CA, AB at E, F respectively.

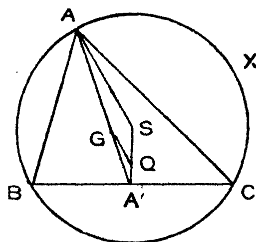
Then $\angle BOD = 90° - \angle OBD = 90° - \angle EBC = \angle C$;

similarly, $\angle COD = \angle B$.

$\therefore$ $\angle BOC = \angle B + \angle C = 180° - \angle A$, and $\therefore$ known.

Hence, the locus of O is a circle passing through B and C and having its centre on the ⊥. bisector of BC. Q. E. D.

2. *Given the base and the vertical angle of a triangle, to find the locus of the centroid of the triangle.*



Let ABC be any Δ standing on the given base BC and having the vertical ∠BAC of a given magnitude.

It is reqd. to find the locus of G, *the centroid of the* Δ ABC.

Proof. Let A′ be the mid-pt. of BC, and G the centroid; then G is a pt. on AA′ such that AA′ = 3GA′.

Let S be the circum-centre; join SA′, SA.

Draw GQ ∥ to AS, meeting SA′ in Q.

Now, since AS is par^l to GQ and AA′ = 3A′G,

$\therefore$ AS = 3GQ.

$\therefore$ GQ = $\frac{1}{3}$AS.

Let XBC be *any other* position of the Δ, the base BC remaining fixed and the vertical ∠BXC being = the ∠BAC.

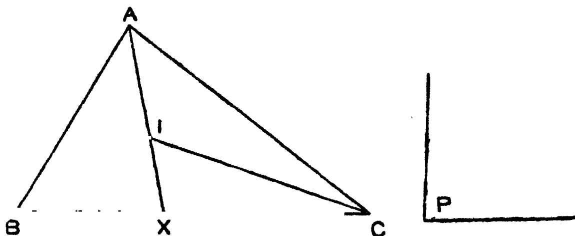
Then, X must be on the arc BAC of the circum-⊙ of the Δ ABC.

Now, if G′ be the centroid of the $\triangle$XBC, it can be shown, as before, that G′Q = $\frac{1}{3}$XS.

Thus, GQ = G′Q (each of them being = $\frac{1}{3}$SA), which shows that G and G′ are on the O^ce of a ⊙ of which the centre is the pt. Q, and the radius = $\frac{1}{3}$SA.

Thus, the reqd. locus is a circle whose centre is Q and radius equal to $\frac{1}{3}$SA. Q. E. D.

3. *Given the base and the vertical angle of a triangle, to find the locus of its in-centre.*



Let ABC be any triangle standing on the given base BC and having the vertical $\angle$BAC = a given $\angle$P; also, let I be the in-centre of the $\triangle$ABC. Join BI, CI and AI; produce AI to meet BC at X.

It is required to find the locus of I.

Proof. The $\angle$BIC = the $\angle$BIX + the $\angle$CIX.

Now, the $\angle$BIX = $\frac{1}{2}$A + $\frac{1}{2}$B; and the $\angle$CIX = $\frac{1}{2}$A + $\frac{1}{2}$C;

$\therefore$ the $\angle$BIC = $\frac{1}{2}$A + $\frac{1}{2}$B + $\frac{1}{2}$A + $\frac{1}{2}$C = A + $\frac{1}{2}$(B + C)

= A + $\frac{1}{2}$(180° − A)

= 90° + $\frac{1}{2}$A = a constant angle, for

the $\angle$A = the $\angle$P = a constant angle.

$\therefore$ the $\triangle$BIC stands on a fixed base and has a constant vertical angle.

$\therefore$ the locus of I is a circle passing through B and C and having its centre lying on the perp. bisector of BC.

Q. E. D.

EXERCISE 20

1. Given the base and the sum of the base angles of a triangle, find the locus of its ortho-centre.

2. Find the locus of the centroid of a triangle which stands on a given base and has the sum of its base angles equal to a given angle.

3. Given the base and the sum of the base angles of a triangle, find the locus of its in-centre.

4. Given the base and the vertical angle of a triangle, find the locus of its ex-centre which is opposite to its vertex.

5. Show that the locus of the middle points of a system of chords of a circle drawn through a fixed point, outside the circle, is another circle intersecting the given one.

6. Show in Ex. 5 that the locus will touch, or will not cut, the given circle according as the fixed point is on, or within the given circle.

7. Show that the points of contact of tangents to a series of concentric circles drawn from a given point outside the circles all lie on a circle.

8. BAC is any triangle described on the fixed base BC and having a constant vertical angle; and BA is produced to P, so that BP is equal to the sum of the sides containing the vertical angle. Find the locus of P.

BOOK IV

THEOREMS ON RECTANGLE AND SQUARE IN RELATION TO THE SEGMENTS OF A STRAIGHT LINE

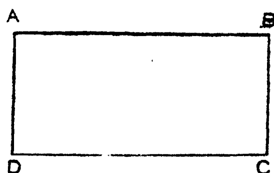
CHAPTER I

GEOMETRICAL THEOREMS CORRESPONDING TO ALGEBRAICAL IDENTITIES

DEFINITIONS AND NOTATIONS

1. A rectangle, of which the adjacent sides are AB and BC, is said to be the **rectangle contained by** AB **and** BC because on these sides alone, its size and shape depend.

The *"rectangle contained by* AB *and* BC" is written as *"rect.* AB, BC", or, more briefly as "AB.BC, or, as *rect.* AC." Likewise, "the *square* described on a straight line AB" is briefly written as *"sq. on* AB," or as AB^2.



2. If D be a point on a straight line, AB, and *within* its extremities, then D is said to *divide* AB **internally,** the parts AD and DB, being called the **segments** of the line AB.

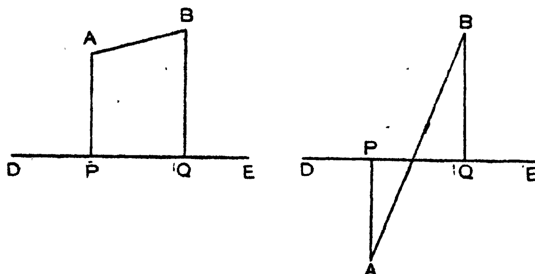


Similarly, if D be on the straight line AB, *produced,* it is said to *divide* AB **externally,** AD and DB being called the **segments** of the line AB.

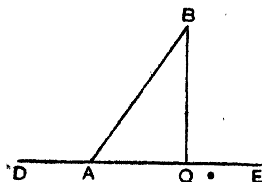


Thus, it is clear that when D is a point of **internal division,** AB is equal to the *sum* of the segments; whereas in case of D being a point of **external division,** AB is equal to the **difference** of the segments.

3. If from the extremities of a straight line AB, perpendiculars AP and BQ be drawn to an unlimited straight line DE, then PQ is said to be the **projection** of AB on DE.



NOTE. If A is on the straight line DE, then AQ becomes the projection.

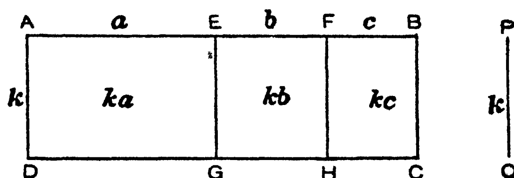


CHAPTER II

RECTANGLES AND SQUARES

THEOREM 49.

If there be two straight lines one of which is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and several parts of the divided line.



Let AB and PQ be two straight lines, of which AB is divided into any number of parts, say three, *viz.* AE, EF, FB; and let PQ, AE, EF, FB contain k, a, b, c units of length respectively; then AB contains $a+b+c$ units.

It is required to prove that

rect. AB, PQ = *rect.* AE, PQ + *rect.* EF, PQ + *rect.* FB, PQ

or, AB.PQ = AE.PQ + EF.PQ + FB.PQ,

i.e. to prove that $(a+b+c)k = ak+bk+ck$.

Through A, draw AD ⊥ to AB; from it cut off AD = PQ; complete the rectangle DABC having AB and AD as adjacent sides.

Draw EG, FH par[l] to AD meeting DC at G and H respectively.

∴ FH = EG = AD = PQ.

Proof. The *fig.* AC = the *fig.* AG + the *fig.* EH + the *fig.* FC...(1)

But, by construction,

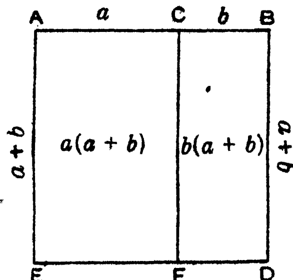
fig. AC = *rect.* AB, AD
i.e. = AB.PQ and contains $(a+b+c)k$ units of area;
fig. AG = *rect.* AE, AD = *rect.* AE, PQ
i.e. = AE.PQ and contains ak units of area;
fig. EH = *rect.* EF, EG = *rect.* EF, PQ
i.e. = EF.PQ and contains bk units of area;
fig. FC = *rect.* FB, FH = *rect.* FB, PQ
i.e. = FB.PQ and contains ck units of area.

Hence, from (1),

rect. AB, PQ = *rect.* AE, PQ + *rect.* EF, PQ + *rect.* FB, PQ
or, AB.PQ = AE.PQ + EF.PQ + FB.PQ
i.e., $(a+b+c)k = ak + bk + ck.$

Q. E. D.

COR. *If a straight line be divided internally into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.*



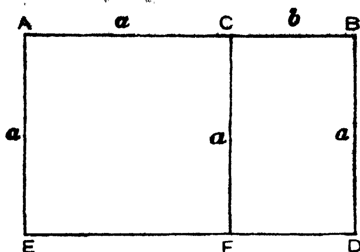
Let AB be the str. line, divided internally at C into AC and CB, which contain a and b units of length respectively.

Describe the sq. ABDE on AB and draw CF, through C, parallel to AE or BD, meeting DE at F.

Proof. Evidently, the figures AF, CD are each a rectangle,

∴ *the sq. on* AB = *rect.* AC, AE + *rect.* CB, CF
= AC. AB + CB.AB [∵ CF = AE = AB]
or, AB^2 = AC.AB + CB.AB,
i.e., $(a+b)^2 = a(a+b) + b(a+b).$

COR. 2. *If a straight line be divided internally into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part together with the rectangle contained by the two parts.*



Let AB be the str. line, divided internally at C, into two parts AC, CB, containing a and b units of length respectively. Describe a square ACFE on AC; draw BD ⊥ to AB, meeting EF produced at D.

Then, the fig. CD = the rect. CF, CB
= the rect. AC, CB and contains ab units of area;

the fig. AF = the sq. on AC and „ a^2 „ „ „ ;
and the fig. AD = the rect. AE, AB
= „ „ AC, AB and „ $a(a+b)$ „ „ .

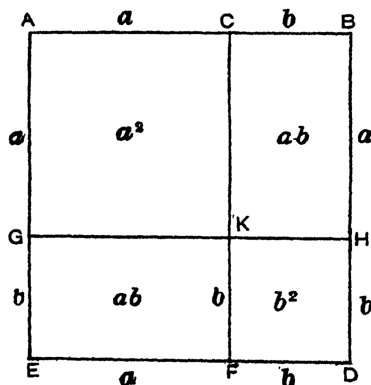
Hence, ∵ the fig. AD = the fig. AF + the fig. CD,

$$AC.\ AB = AC^2 + AC.CB$$

i.e., $a(a+b) = a^2 + ab.$

THEOREM 50.

If a straight line be divided **internally** *at any point, the square on the whole line is equal to the sum of the squares on the two parts* **together with** *twice the rectangle contained by the parts.*



Let AB be a str. line which is divided internally, at C; and let the parts AC, CB contain a and b units of length respectively, so that AB contains $a+b$ units.

It is required to prove that

$$AB^2 = AC^2 + CB^2 + 2AC.CB,$$

$$i.e., \ (a+b)^2 = a^2 + b^2 + 2ab.$$

Suppose ABDE to be the square described on AB; from AE cut off AG = AC or a, so that GE(= CB) contains b units of length. Through C and G, draw CF and GH, parallel respectively to AE and AB; and let CF and GH intersect at K. Then, evidently HD = KF = GE = FD = KH = CB, each containing b units of length.

Proof. The fig. AD = the fig. AK + the fig. KD + the fig. CH + the fig. GF......(1)

By construction,

the fig. $AD = AB^2$, containing $(a+b)^2$ units of area;
" " $AK = AC^2$, " a^2 " " ";
" " $KD = KH^2 = CB^2$, " b^2 " " ";
" " $CH = CK.CB$
" " $= AC.CB$ " ab " " ";
and " " $GF = GK.GE$
$= AC.CB$, " ab " " ".

Hence, from (1), $AB^2 = AC^2 + CB^2 + AC.CB + AC.CB$
$= AC^2 + CB^2 + 2AC.CB$,
i.e. $(a+b)^2 = a^2 + b^2 + 2ab$.

COR. *If a str. line be bisected, the square on half the line is equal to one-fourth of the square on the whole.*

EXERCISE 1

(Numerical.)

1. Establish the following identities geometrically :—
   (i) $(3+5)7 = 3 \times 7 + 5 \times 7$.
   (ii) $5(5+9) = 5^2 + 5 \times 9$.
   (iii) $(3+4)^2 = 3^2 + 2 \times 3 \times 4 + 4^2$.

2. The length and breadth of a garden are respectively 100 ft. and 75 ft.; if the garden be divided into two equal parts, lengthwise, find geometrically, the area of each.

THEOREM 51.

If a straight line be divided **externally** *at any point, the square on the line is equal to the sum of the squares on the two parts* **diminished** *by twice the rectangle contained by the parts.*

Let AB be the straight line which is divided *externally* at C; and let the parts AC, CB contain a and b units of length respectively; then AB contains $a-b$ units.

It is required to prove that

$$AB^2 = AC^2 + CB^2 - 2AC.CB,$$

i.e., $(a-b)^2 = a^2 + b^2 - 2ab.$

Suppose ABDE to be the square described on AB; from AE produced, cut off $EG = CB = b$; through C and G, draw CK, GK, par^l to AG, AC respectively, intersecting each other at K; produce BD to meet GK, at H. Also produce DE, HG to L and M respectively, making $EL = GM = EG = CB$, or b. Then, $DL = DE + EL = AB + BC = AC$, or a; and similarly, $HM = AC$, or a.

Proof. The fig. AD = the fig. AK + the fig. ME
− the fig. MD − the fig. HC......(1)

By construction,

fig. AD = *sq. on* AB = AB^2 and contains $(a-b)^2$ units of area;
fig. AK = *sq. on* AC = AC^2 " " a^2 " " " ;
fig. ME = *sq. on* EG
= *sq. on* CB = CB^2 " " b^2 " " " ;
fig. MD = *rect.* DL, LM
= *rect.* AC, CB = AC.CB and contains ab " " " ;
fig. HC = *rect.* HB, CB = AC.CB " " ab " " " .

Hence, from (1), $AB^2 = AC^2 + CB^2 - AC.CB - AC.CB$
$= AC^2 + CB^2 - 2AC.CB$,
i.e., $(a-b)^2 = a^2 + b^2 - 2ab$.

COR. 1. *The square on the sum of two straight lines together with the square on their difference is equal to twice the sum of the squares on the given straight lines.*

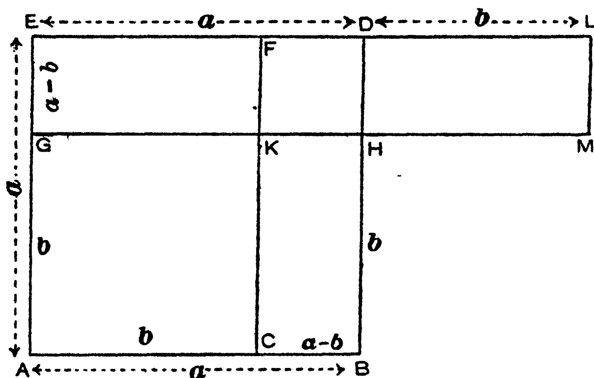
COR. 2. *The square on the sum of two str. lines exceeds the square on their difference by four times the rectangle contained by the lines.*

Obs. The two enunciations of Theorems 50 and 51 may be combined into a single one as :

"*If a straight line be divided at a point, the square on the given line is equal to the sum of the squares on the two parts, so formed,* ***increased*** *or* ***diminished*** *by twice the rectangle contained by the parts according as the point of division is* ***internal*** *or* ***external.***"

THEOREM 52.

The difference of the squares on any two straight lines is equal to the rectangle contained by their sum and difference.



Let two str. lines AB and AC (AB being greater than AC) be so placed, *one along the other*, that one extremity of the one is coincident with one extremity of the other; and let them contain a and b units of length respectively.

It is required to prove that

$$AB^2 - AC^2 = (AB + AC)(AB - AC).$$

i.e. $\quad a^2 - b^2 = (a+b)(a-b).$

Suppose ABDE to be the square described on AB; from AE, cut off AG = AC or b; through G and C draw GH, CK par^ls to AB and AE respectively, intersecting each other at K. Produce GK to meet BD at H. Produce ED to L, making DL = AC or b. Complete the rectangle HDLM. Then, evidently, GE = AE − AG = AB − AC = CB or $a - b$; also, the fig. AK = the sq. on AC.

Proof. The rect. HL = DL.LM = AC.CB;
also the rect. CH = CK.KH = AC.CB.
∴ the rect. HL = the rect. CH.

$AB^2 - AC^2$ = the fig. AD − the fig. AK
= the rect. EH + the rect. CH
= the rect. EH + the rect. HL = the rect. EM
= EL.EG = (ED + DL)(AE − AG)
= (AB + AC)(AB − AC),

i.e. $a^2 - b^2 = (a+b)(a-b)$.

COR. 1. *If a straight line be bisected, and also divided into two unequal segments either internally or externally then the rectangle contained by the unequal segments is equal to the difference of the squares on half the line and on the line between the points of section.*



Fig. 1.



Fig. 2.

For, if the str. line AB be bisected at C and also divided into two unequal segments at D, then in fig. (1), we have

AD.DB = (AC + CD)(CB − CD)
= (AC + CD)(AC − CD), ∵ AC = CB;
= $AC^2 - CD^2$;

and in fig. (2),

AD.DB = (CD + AC)(CD − CB)
= (CD + AC)(CD − AC), ∵ AC = CB;
= $CD^2 - AC^2$.

COR. 2. *If a straight line be bisected, and also divided into two unequal segments, either internally or externally, the difference of the squares on the two segments is equal to twice the rectangle contained by the whole line and the line between the points of section.*

In fig. 1, *Cor. 1,* $AD^2 - DB^2$ = (AD + DB)(AD − DB)
= AB(AD − DB)

Now $AD = AC + CD$
and $DB = CB - CD$
$\therefore$ $AD - DB = 2CD$, $\because$ $AC = CB$.
$\therefore$ $AD^2 - DB^2 = AB.2CD = 2AB.CD$.

Similarly, in fig. 2, *Cor. 1*, since
$$AD + DB = (AC + CD) + (CD - CB).$$
$$= 2CD, [\because AC = CB]$$
$$AD^2 - DB^2 = (AD + DB)(AD - DB)$$
$$= 2CD.AB = 2AB.CD.$$

EXERCISE 2

1. Illustrate and explain the geometrical theorem corresponding to the identity
$$(2a)^2 = 4a^2.$$

2. Illustrate and explain the geometrical theorem corresponding to the identity
$$(3a)^2 = 9a^2.$$

3. Prove geometrically that
$$(a+b)(c+d) = ac + ad + bc + bd.$$

4. Prove geometrically that
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$$

5. Prove geometrically that
$$(a-b)(a-c) = a^2 - ac + bc - ab.$$

6. Illustrate and explain the geometrical theorem corresponding to the identity
$$(a+b)(b+c) = ac + b(a+b+c).$$

7. If AD be the perpendicular to the hypotenuse BC of a right-angled triangle ABC, prove that
$$AD^2 = BD.DC.$$

Hence prove that (i) $AB^2 = BC.BD$, and (ii) $AC^2 = BC.CD$.

8. ABC is a triangle, and AD is perpendicular to BC. If $AD^2 = BD.DC$, prove that $\angle BAC$ is a right angle.

9. C is any point in a given straight line AB. Deduce from Cor. 2, Th. 49 and Th. 50 that
$$AB^2 + BC^2 = AC^2 + 2AB.BC.$$

10. If C be the mid-point of a given straight line AB, and if D is any point in CB, prove that

$$AD^2+DB^2=2AC^2+2CD^2.$$

11. If C is the mid-point of a given straight line AB, and if D is any point in CB produced, prove that

$$AD^2+DB^2=2AC^2+2CD^2.$$

12. C is the mid-point of a given straight line AB. If a point D moves from B to A, prove that the area of the rect. AD.DB gradually *increases* until D coincides with C, when it has its *greatest* value, and thereafter gradually *diminishes*.

13. In the previous case, prove that AD^2+DB^2 gradually *diminishes* until B coincides with C, when it has its *least* value, and thereafter gradually *increases*.

Hence show that if the sum of two straight lines be given, the least value of the sum of the squares on the lines is equal to half the square on the given sum.

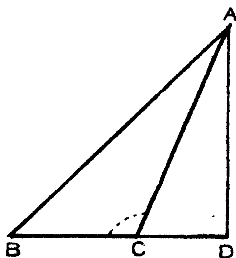
14. AB is a given straight line. If a point C moves from B to A, prove that AB^2+BC^2 is always *greater than* 2AB.BC, except when C coincides with A; and that in this case $AB^2+BC^2=2AB.BC$.

CHAPTER III

RECTANGLES AND SQUARES IN RELATION TO TRIANGLES

THEOREM 53.

In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the other two sides together with twice the rectangle contained by one of these two sides and the projection on it of the other.



Let ABC be a $\triangle$, obtuse-angled at C; and let AD be $\perp$ to BC produced, so that CD is the projection of AC on BC.

It is required to prove that

$$AB^2 = BC^2 + CA^2 + 2BC.CD.$$

Proof. $BD^2 = (BC + CD)^2 = BC^2 + CD^2 + 2BC.CD.$ *Th. 50.*

To each of these equals add DA^2.

Then, $BD^2 + DA^2 = BC^2 + CD^2 + DA^2 + 2BC.CD.$

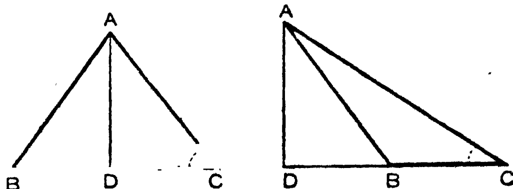
But $\left.\begin{array}{l} BD^2 + DA^2 = AB^2, \\ CD^2 + DA^2 = AC^2. \end{array}\right\}$ *Th. 31.*
(and)

$\therefore \quad AB^2 = BC^2 + CA^2 + 2BC.CD.$

Q. E. D.

THEOREM 54.

In any triangle the square on the sides opposite to an acute angle is equal to the sum of the squares on the other two sides diminished by twice the rectangle contained by one of these two sides and the projection on it of the other.



Let ABC be any $\triangle$, acute-angled at C; and let AD be $\perp$ to CB, or CB produced, so that CD is the projection of AC on BC.

It is required to prove that

$$AB^2 = BC^2 + CA^2 - 2BC.CD.$$

Proof. Since BD is the difference of the two str. lines BC, CD, we have

$$BD^2 = BC^2 + CD^2 - 2BC.CD. \qquad \textit{Th. 51.}$$

To each of these equals add DA^2.

Then $BD^2 + DA^2 = BC^2 + CD^2 + DA^2 - 2BC.CD.$

But $BD^2 + DA^2 = AB^2$, and $CD^2 + DA^2 = AC^2$, *Th. 31.*

$$\therefore \; AB^2 = BC^2 + CA^2 - 2BC.CD.$$

Q. E. D.

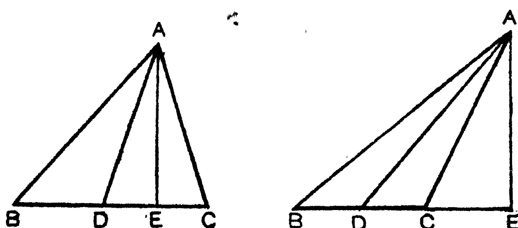
NOTE. Comparing Theorems 31, 53 and 54, we come to the following conclusion :—

The square on a side of a triangle is greater than, equal to, or less than, the sum of the squares on the other two sides, according as the angle contained by those sides is obtuse, right, or acute ; the difference in the cases of inequality being twice the rectangle contained by one of the two sides and the projection on it of the other.

Apollonius' Theorem.

THEOREM 55.

The sum of the squares on any two sides of a triangle is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.



Let AD be the median which bisects the side BC of a $\triangle$ABC.

It is required to prove that $AB^2 + AC^2 = 2BD^2 + 2DA^2$.

Proof. Let AE be $\perp$ to BC, or BC produced; then in the above diagrams, if the $\angle$ADB is obtuse, then the $\angle$ADC is acute.

Hence in the $\triangle$ ABD,

$AB^2 = BD^2 + DA^2 + 2BD.DE$; *Th. 53.*

and in the $\triangle$ADC,

$AC^2 = CD^2 + DA^2 - 2CD.DE$ *Th. 54.*

$= BD^2 + DA^2 - 2BD.DE$, $\because$ CD = BD.

Hence by addition,

$AB^2 + AC^2 = 2BD^2 + 2DA^2$.

Q. E. D.

Pappus' Theorem.

THEOREM 56.

If a straight line is drawn from the vertex of an isosceles triangle to any point in the base, or the base produced, the difference between the squares on this line and one of the equal sides of the triangle is equal to the rectangle contained by the segments of the base.

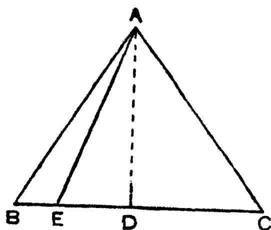


Fig. 1.

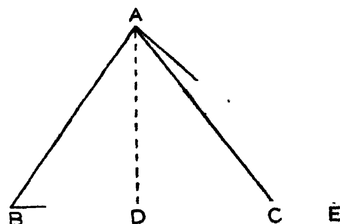


Fig. 2.

Let ABC be an isos. Δ and E be a point on the base BC, or BC produced.

It is required to prove that $AB^2 \sim AE^2 = BE.EC$.

From A draw AD $\perp$ to BC.

Proof. $AB^2 = AD^2 + BD^2$ and $AE^2 = AD^2 + DE^2$.

$\therefore$ if E be an internal point as in fig. 1,

$$AB^2 - AE^2 = BD^2 - DE^2 = (BD - DE)(BD + DE)$$
$$= BE(CD + DE). \qquad [\because BD = CD]$$
$$= BE.CE.$$

Again, if E be an external point as in fig. 2,

$$AE^2 - AB^2 = DE^2 - BD^2 = (DE + BD)(DE - BD)$$
$$= BE(DE - CD)$$
$$= BE.CE.$$

COR. 1. *The difference of the squares on the two sides of any triangle is equal to twice the rectangle contained by the base and the projection on the base of the median to it.*

EXERCISE 3

1. The numerical measures of the sides of a triangle are 7, 9 and 11; prove that the angle opposite to the greatest side is neither right nor obtuse.

Construct the triangle, taking one-tenth of an inch as the unit of length.

2. The numerical measures of the sides of a triangle are 8, 13 and 17; show that the angle, opposite to the greatest side must be obtuse.

Construct the triangle, taking three-tenths of a centimetre as the unit of length.

3. ABC is an acute-angled triangle. The perpendiculars from A, B, C upon the opposite sides meet at O. Prove that

$$OA^2+OB^2+OC^2<\tfrac{1}{2}(AB^2+BC^2+CA^2).$$

4. The numerical measure of a side of an equilateral triangle ABC is a. BC is produced to D so that CD=2BC. If x be the numerical measure of AD, prove that $x^2=7a^2$.

5. ABC is an acute-angled triangle; and BE, CF are perpendiculars upon CA, AB. Prove that AB.AF=AC.AE.

6. D is the mid-point of the side BC of a triangle ABC; and AE is perpendicular to BC, or BC produced. Use Theorems 53 and 54 to prove that the difference of the squares on AB, AC is equal to twice the rect. BC.DE.

7. C is the mid-point of a given straight line AB. If a point P moves on the circumference of a circle of which the centre is C, prove that PA^2+PB^2 is always the same.

8. The distance between two fixed points A and B is 6 inches. If a point P moves so that PA^2+PB^2 is always=50 square inches, find the locus of P.

9. Prove that the sum of the squares on the two equal sides of an isosceles triangle is less than the sum of the squares on the two sides of any other triangle on the same base and between the same parallels.

10. ABCD is a rectangle; and P is any point, either inside or outside it. Prove that $PA^2+PC^2=PB^2+PD^2$.

11. Prove that three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

12. If O is the centroid of a $\triangle ABC$, prove that

$$3(OA^2+OB^2+OC^2)=AB^2+BC^2+CA^2.$$

13. Prove that the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on its diagonals together with four times the square on the line joining the middle points of the diagonals.

Hence prove that if the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals, the quadrilateral must be a parallelogram.

CHAPTER IV

RECTANGLES AND SQUARES IN RELATION TO CIRCLES

THEOREM 57.

If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

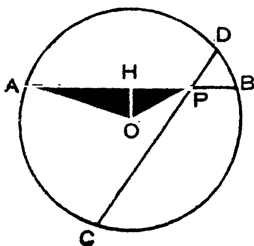


Fig. 1.

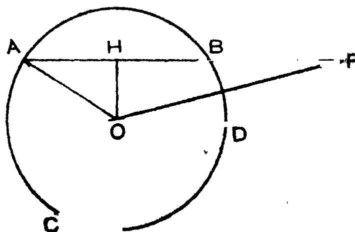


Fig. 2.

Let the chords AB, CD of the ⊙, whose centre is O, intersect inside or outside the ⊙ at P.

It is required to prove that AP.PB = CP.PD.

Proof. Let OH be the ⊥ from O upon AB.

Join OP, OA.

Since OH is ⊥ to AB, ∴ AH = HB.

(i) When P is inside the ⊙, as in fig. (1),

$AP.PB = (AH + HP)(BH - HP)$

$= (AH + HP)(AH - HP), \quad \because \quad AH = BH.$

$= AH^2 - HP^2$

$= (AH^2 + HO^2) - (HP^2 + HO^2)$, adding and subtracting HO^2,

$= OA^2 - OP^2.$

Similarly, $CP.PD = OC^2 - OP^2.$

Hence, $\because \quad OA = OC$, *i.e.* $OA^2 = OC^2$,

we have, $AP.PB = CP.PD.$

(ii) When P is outside the ⊙, as in fig. (2),

$AP.PB = (AH + HP)(HP - HB)$

$= (HP + AH)(HP - AH), \quad \because \quad AH = HB;$

$= HP^2 - AH^2$

$= (HP^2 + OH^2) - (AH^2 + OH^2)$, adding and subtracting OH^2,

$= OP^2 - OA^2.$

Similarly, $PC.PD = OP^2 - OC^2.$

Since $OA^2 = OC^2$,

$\therefore \quad AP.PB = PC.PD.$

Q. E. D.

THEOREM 58.

[CONVERSE OF THEOREM 57.]

If two finite straight lines intersect, either ***both internally*** *or* ***both externally****, so that the rectangle contained by the segments of the one is equal to that contained by the segments of the other, the extremities of the lines are concyclic.*

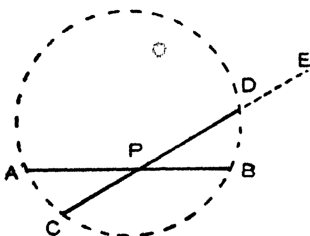


Fig. 1.

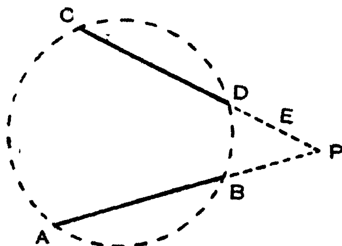


Fig. 2.

Let AB and CD be two str. lines which intersect, internally as in fig. 1, or externally as in fig. 2, at P; and let AP.PB = CP.CD.

It is required to prove that the pts. A, B, C, D *are concyclic.*

Proof. If not suppose, the circle drawn through A, B, C cuts CD, or CD produced, at E.

Then, AP.PB = CP.PE; *Th. 57.*

but AP.PB = CP.PD *Hyp.*

∴ CP.PD = CP.PE

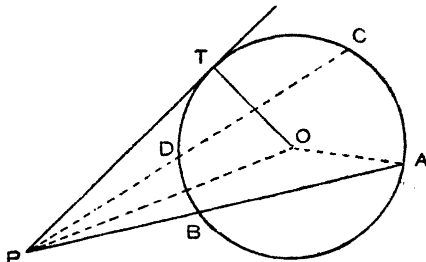
∴ PD = PE, which is impossible unless D and E coincide.

Hence, A. B, C, D lie on the same circle.

Q. E. D.

THEOREM 59.

If a tangent and a secant be drawn to a circle from a point outside it, the square on the tangent is equal to the rectangle contained by the whole secant and the part of it outside the circle.



Let ABC be a ⊙ whose centre is O; from the point P, outside the ⊙, let PT be drawn a tangent touching the ⊙ at T and PBA, a secant cutting the ⊙ at B and A.

It is required to prove that $PT^2 = PA.PB$.

Join OP, OA, OT.

Proof. $PA.PB = OP^2 - OA^2$ *Th. 57.*

$= OP^2 - OT^2$, ∵ $OA = OT$

$= PT^2$, ∵ OT is ⊥ to PT.

Q. E. D.

By the method of Limits.

Let the secant PDC cut the ⊙ in D and C. Suppose PC to be turned about P so that the pts. C and D continually approach each other.

In every position of the secant PC, we have $PA.PB = PC.PD$; hence, ultimately, when the pts. C and D coincide at T and the secant PDC becomes the tangent PT, we must have $PA.PB = PT^2$.

COR. *The two tangents drawn, one to each, of two intersecting circles from any point on their common chord produced are equal.*

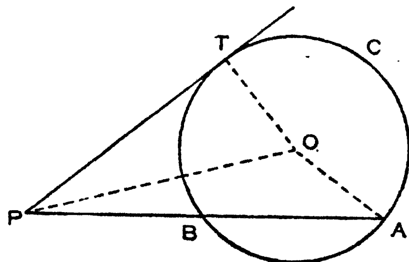
For, if P be any point on the common chord AB produced, and if PS and PT be the two tangents, $PT^2 = PA.PB = PS^2$.

∴ $PT = PS$.

THEOREM 60.

[CONVERSE OF THEOREM 59.]

If from a point without a circle, there be drawn a secant to cut, and another line to meet, the circle, and if the rectangle contained by the whole secant and the part of it outside the circle, be equal to the square on the other line, then the latter is a tangent to the circle.



Let ABC be a ⊙ whose centre is O ; and from a point, P outside the ⊙, let PBA be the secant cutting the ⊙ at B and A, and PT, another str. line meeting the ⊙ at T, such that $PA.PB = PT^2$.

It is required to prove that PT *touches the* ⊙ *at* T.

Proof. Join OP, OA, OT.

$PA.PB = OP^2 - OA^2$ *Th. 57.*

$= OP^2 - OT^2$, ∵ OA = OT, *being radii of the same* ⊙.

But $PA.PB = PT^2$.

∴ $OP^2 - OT^2 = YT^2$,

or $OP^2 = OT^2 + PT^2$.

∴ the ∠OTP is a rt. ∠.

Hence, PT touches the ⊙ at T.

SECOND METHOD.

If PT do not touch the ⊙ at T, let PT produced cut the ⊙ again in some other pt. S. Then PT.PS = PA.PB; but $PA.PB = PT^2$; ∴ $PT.PS = PT^2$, which is impossible unless S and T coincide.

Hence PT cuts the ⊙ in two coincident points at T *i.e.* touches ⊙ at T.

EXERCISE 4

1. Chords AB and CD of a circle intersect at P. If AP=6″, CP=8″ and DP=3″, find BP.

2. Chords AB and CD of a circle intersect externally at P. If PA=8″, PB=3″, PC=6″, find the length of PD.

3. From a point P, outside a circle, a secant PBA is drawn to cut the circle at B and A. If PT be the tangent to the circle and PT=6″, PB=4″, find the length of AB.

4. A chord AB passes through a given point P within a circle whose centre is O and radius is 5″. If PA=8″. PB=2″, show that the length of the chord bisected at P is 8″.

5. Show that the tangents, drawn from any point on the common chord produced, to two intersecting circles, are equal.

6. The rectangle under the segments of a chord through a given point within a circle is equal to the square on half the chord bisected at that point.

7. If from the vertex A of a right-angled triangle ABC, right-angled at A, a perpendicular AP be drawn to the base BC, show that (i) $BP.PC = AP^2$, (ii) $BP.BC = BA^2$ and (iii) $CP.CB = CA^2$.

8. ABC is a triangle in which AX, BY, CZ are the perpendiculars from the vertices to the opposite sides. If the perpendiculars meet at O, prove that

$$AO.OX = BO.OY = CO.OZ.$$

9. A, B, C are three points on a straight line. Find the locus of points of contact of tangents drawn from A to the circles passing through B and C.

10. Show that the points from which tangents to two given intersecting circles are equal lie on their common chord.

11. Show how to draw from any point P on a chord AB, a str. line to meet the circumference at R so that $PR^2=PA.PB$.

12. Prove that each of the common tangents to two intersecting circles is bisected by their common chord produced.

MISCELLANEOUS EXERCISES

1. Draw figures to illustrate the following identities :

 (i) $a(b-c)=ab-ac$;

 (ii) $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$;

 (iii) $(a+b)(x+y)=ax+ay+bx+by$.

2. State and prove the geometrical theorem corresponding to the identity

$$(a+b)^2-(a-b)^2=4ab.$$

3. State and prove the geometrical theorem corresponding to the identity

$$(a+b)^2+(a-b)^2=2(a^2+b^2).$$

4. If a straight line AB be bisected at C and P be any other point in AB, or AB produced, then $PA^2+PB^2=2PC^2+2CA^2$.

5. Four points A, B, C, D are taken in this order on a straight line ; show that

$$AD^2+BC^2=AC^2+BD^2+2AB.CD.$$

6. C is the middle point of a straight line AB and D is any point in AB, or in AB produced. If $AD>BD$, prove that

$$AD^2-BD^2=2CD.AB.$$

7. D is the middle point of the side AB of the triangle ABC, and CN is the perpendicular from C to AB. If $AC>BC$, prove that

$$AC^2-BC^2=2AB.DN.$$

8. D is any point on the base BC produced, of a triangle ABC, in which $AB=AC$; prove that

$$AD^2=AB^2+BD.CD.$$

9. A straight line AB is divided at C so that $AB.BC=AC^2$. Prove that

$$AB^2+BC^2=3AC^2.$$

10. Divide a given straight line into two parts such that the rectangle contained by them may be the greatest possible.

11. Prove that of all rectangles which have a given perimeter, one of the greatest area is a square.

12. Explain how to find a point Y in AB produced, such that the rect. AY.YB may be equal to a given square one of whose sides is equal to c.

[Bisect AB at C. Draw BD ⊥ to AB and $=c$. With centre C and radius CD, draw a ⊙ to cut AB in Y.]

13. Find a point X in a given straight line AB, or in AB produced, such that the sum of the squares on AX, XB may be equal to a given square whose side is c.

[Draw a str. line BY at B making $\angle ABY=45°$; with centre A and radius c draw a circle to cut BY in Y and Y′; the foot of the perp. from Y or Y′ is X.]

14. Find an expression for the length of the perpendicular drawn from the vertex of a triangle to its base in terms of the lengths of its sides.

15. Find the area of the triangle whose sides are 10″, 17″ and 21″.

16. Calculate the length of the median drawn to the longest side of a triangle whose sides are 7″, 24″ and 25″.

17. Find the length of the base of a triangle whose sides are 7 cm. and 9 cm. and the length of the median bisecting the base is 8 cm.

18. Prove that the sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on the sides.

19. Show that in any right-angled triangle the sum of the squares on the medians is equal to one and a half times the square on the hypotenuse.

20. If P, Q be the points of trisection of the base BC of a triangle ABC, prove that

$$AB^2+AC^2=AP^2+AQ^2+4PQ^2.$$

21. The chord of an arc of a circle, whose diameter is d, is c and the height of the arc is h. Show that $4h(d-h)=c^2$.

[*Definition :* The height of an arc of a circle is the straight line joining the middle points of the arc to the middle point of the chord on which the arc stands.]

22. In a circle of radius r, the chord of an arc is $2m$ and the height of the arc is h. Show that $r=\frac{m^2+h^2}{2h}$.

23. In a circle of radius r, the chord of half an arc is m and the height of the arc is h. Show that $r=\frac{1}{2}\frac{m^2}{h}$.

24. If the radius of the circular arch of a bridge be 32 ft. and the height of the bridge be 12 ft., show that the span of the bridge is very nearly 50 ft.

25. A brick 4″ thick is placed so as to block a carriage wheel. If the distance of the brick from the point of contact of the wheel and the ground is 10′, find the radius of the wheel.

26. If d denotes the shortest distance from an external point to a circle and t the length of the tangent from the same point, show that $d(d+2r)=t^2$.

27. C is the centre of a given circle and PQ is any chord through a fixed point O, within the circle. The tangents at P and Q meet at T. Prove that T lies on a fixed str. line perpendicular to CO. [Draw TN ⊥ to CO. Join CT meeting PQ at M. Prove that CO.CN = CT.CM = CP^2.]

28. If in Ex. 27, CT meets PQ at M and TN be the perpendicular on CP, show that the points O, M, T, N lie on a circle.

29. AB is a chord of a circle whose centre is C and tangents at A, B meet in T. Join CT cutting AB at N. Through N draw any chord PNQ. Prove that the points C, P, T, Q are concyclic.

30. AB is the common chord of two given circles; and through any point in AB, two chords are drawn one in each circle. Prove that the four extremities of these chords all lie on a circle.

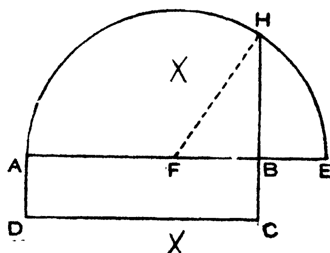
CHAPTER V

PROBLEMS ON AREAS (*Contd.*)

I. On Construction of Squares and Rectangles.

PROBLEM 42.

To construct a square equal in area to a given rectangle.



Let ABCD be the given rectangle, of which the side AB is > the side BC.

It is required to construct a square equal in area to the rectangle AC.

Construction. Produce AB to E, making BE = BC.

Bisect AE at F; with centre F, and radius FA, or FE, describe a semi-circle.

Produce CB to meet the $\bigcirc^{ce}$ in H.

Then the sq. described on BH is the required square.

Proof. Join FH.

Now, since the $\angle$ FBH is a rt. $\angle$.

$\therefore \quad BH^2 = HF^2 - FB^2$ *Th. 45.*

$= AF^2 - FB^2$

$= (AF + FB)(AF - FB)$ *Th. 52.*

$= (AF + FB)(EF - FB)$

$= AB.\ BE$

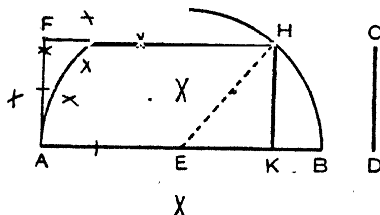
$=$ the rect. AC. ($\because$ BE = BC.) Q. E. F.

COR. *Construct a square equal to a given rectilineal figure.*

All that we have to do is (i) to construct a triangle equal in area to the given rectl. figure; and then (ii) to construct a rectangle equal in area to the triangle and finally (iii) to construct a square equal in area to the rectangle.

PROBLEM 43.

Divide a given straight line internally so that the rectangle contained by the two segments may be equal to a given square, a side of the square being less than half the given straight line.



Let AB be the given straight line, and CD a side of the given square.

It is required to find a pt. K *in* AB *so that the rect.* AK. KB *may be = the sq. on* CD.

Constrnction. Bisect AB at E.

With E as centre and EA as radius, describe a semi-circle

Draw AF ⊥ to AB, and from it cut off AF = CD.

Draw FH ‖ to AB, and let H be one of the points where FH cuts the semi-circle.

Draw HK ⊥ to AB; then K is the reqd. pt.

Proof. Join EH.

FK is a rectangle; ∴ HK = FA = CD.

Now, $AK.KB = AE^2 - EK^2$ *Th. 52, Cor. 1.*

$= EH^2 - EK^2$

$= HK^2$

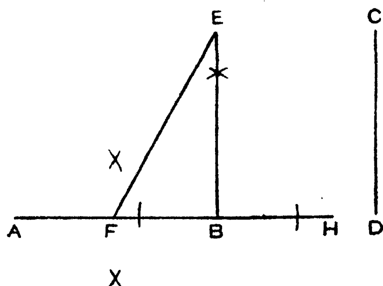
$= CD^2$. Q. E. F.

NOTE. Hence, given the area of a rectangle and the *sum* of its adjacent sides, we know how to construct the rectangle.

If the area be given as equal to that of a given *rectangle*, we have first of all to construct a square equal to this rectangle (Problem 42), and then proceed as above.

PROBLEM 44.

Divide a given straight line externally so that the rectangle contained by the two segments may be equal to a given square.



Let AB be the given str. line, and CD a side of the given square.

It is required to find a point H *in* AB *produced so that the rect.* AH.HB *may be = the sq. on* CD.

Construction. Draw BE ⊥ to AB, and from it cut off BE = CD.

Bisect AB at F, and join FE.
Produce FB to H, making FH = FE.
Then H is the required pt.

Proof. AB is bisected at F and divided *externally* at H ;

$$\therefore \quad AH.HB = FH^2 - FB^2$$
$$= FE^2 - FB^2$$
$$= EB^2$$
$$= CD^2.$$

Q. E. F.

NOTE. Hence, given the area of a rectangle and the *difference* of its adjacent sides, we know how to construct the rectangle.

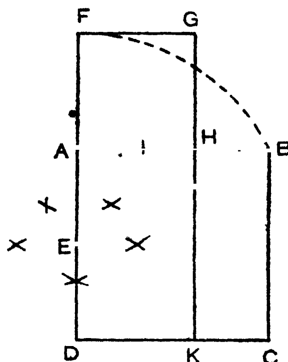
If the area be given as equal to that of a given rectangle, we have first of all to construct a square equal to this rectangle and then proceed as above.

II. Medial Section.

4. Definition : A straight line is divided in **medial section** when the rectangle contained by the whole line and one of the parts is equal to the square on the other part.

PROBLEM 45.

To divide a given straight line ***internally*** *into two segments such that the rectangle contained by the whole line and one of the segments is equal to the square on the other.*



Let AB be the given straight line.

It is required to find an internal point H *in* AB, *so that* AB.BH = AH^2.

Construction. On AB describe the square ABCD.

Bisect AD at E, and join EB.

Produce EA to F, making EF = EB.

From AB cut off AH = AF ; then H is the reqd. pt.

Proof. Through H draw GHK ‖ to FD, meeting DC in K; and draw FG ‖ to AB, meeting KH produced in G.

Now FK is a rectangle, which = DF.FG *i.e.*, = DF.FA; and FH is the sq. on AH.

Since DA is bisected at E and divided *externally* at F,

Cor. 1, Th. 52.

$\therefore$ DF.FA = $EF^2 - EA^2 = EB^2 - EA^2$

$= AB^2$ ($\because$ *the* $\angle$EAB *is right*).

Thus, the rect. FK = the rect. AC.

Taking away the common part AK, we have

the sq. FH = the rect. HC

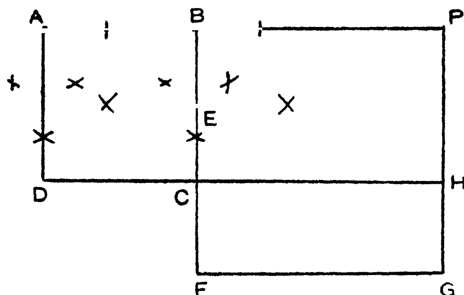
= the rect. AB, BH ($\because$ CB = AB).

That is, AH^2 = AB.BH. Q. E. F.

5. Definition : *The point* H, *so found out, is called the* **point of medial section.**

PROBLEM 46.

To divide a given straight line **externally** *into two segments so that the rectangle contained by the whole line and one of the segments is equal to the square on the other.*



Let AB be the given straight line.

It is required to divide AB *externally at* P, *such that*

$$AB.AP = PB^2.$$

Construction. On AB, describe a square ABCD.

Bisect BC at E and join AE. From BC produced, cut off EF = EA. On BF, describe a square FBPG.

Then, P is the reqd. pt. of medial section.

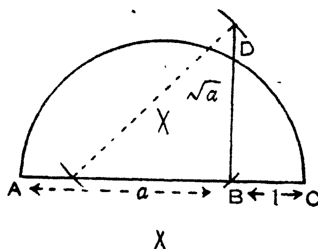
Proof. $FB.FC = (FE+EB)(FE-EC)$
$= (FE+EB)(FE-EB)$. [$\because$ EB = EC.]
$= FE^2 - EB^2$
$= EA^2 - EB^2$
$= AB^2$.

Now, PB^2 = fig. PF = fig. PC + fig. HF
= rect. PB, BC + rect. FC, CH
$= PB.BC + FC.FB$
$= PB.BC + AB^2$
$= PB.BC + AB.BC$
$= AP.BC = AP.AB$. Q. E. F.

CHAPTER VI

EASY APPLICATION OF GEOMETRY TO ALGEBRA.

1. *To extract geometrically the square-root of a number.*



Let a be the given number.

It is required to find the value of $\sqrt{a}$.

Construction. Let AB be a str. line containing a units of length.

Produce AB to C, making BC$=1$ unit of length.

On AC as diameter, describe a semi-circle to meet the perp., through B, to AB at D.

Then, the number representing the length BD is the reqd. sq. root.

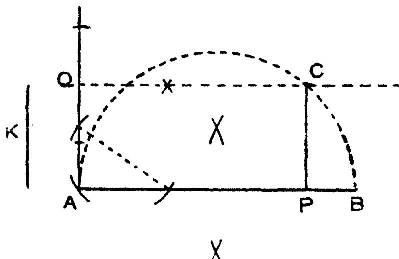
Proof. Evidently, the angle at D, on the semi-circle, subtended by the diameter AC is a right angle ; the $\angle$ADC$=$ a rt. $\angle$; and DB is perp. to the hyp. AC.

$$\therefore \quad BD^2 = AB.BC = a.$$

$$\therefore \quad BD = \sqrt{a}.$$

2. *Solve the equations* $\left.\begin{matrix} x+y=b \\ xy=a \end{matrix}\right\}$

Let AB be a str. line containing b units of length, and K the str. line representing the sq. root of the number a.

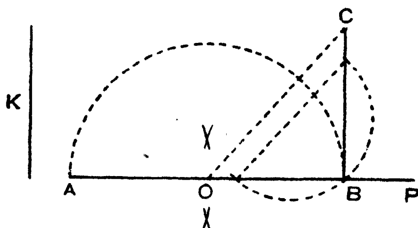


Draw a semi-circle on AB as diameter.

Through A, draw AQ perp. to AB and = K. Through Q draw QC par^l to AB, meeting the ⊙ at C. Then, if P be the foot of the perp. from C to AB, AP and PB represent the roots of the equations. For, AP + PB = AB = b; and AP.PB = $CP^2 = K^2 = a$.

3. *Solve the equations* $\left.\begin{matrix} x-y=b \\ xy=a \end{matrix}\right\}$

Let AB be the given straight line containing b units of length; and K the str. line representing the sq. root of the number a.



Draw a semi-circle on AB as diameter, and let O be its centre. From the perp. BC, to AB, cut off BC = K. Join OC. From OB produced, cut off OP = OC. Then, AP, PB denote x and y respectively.

Proof. PB.PA $= OP^2 - OB^2 = OC^2 - OB^2 = BC^2 = K^2 = a$,
and AP − BP = AB = b.

Obs. Any quadratic equation may be solved with the help of the above two ; for, a quadratic equation can always be reduced to one or other of the forms $a=x(b-x)$ or $a=x(x-b)$; and replacing $b-x$ or, $x-b$ in the above two, by y, the equations reduce to the case of either Art. 2 or Art. 3.

EXERCISE 5

1. Construct a square equal in area to a rectangle whose sides are 8″ and 3″.

2. A rectangle whose length is 5″ is equivalent to a square whose side is 2·5″. Show by geometric construction that its breadth is 1·25″.

3. Show, by geometrical construction, that the numbers, whose sum is 13 and product is 42, are 6 and 7.

4. Solve the equations geometrically (i) $\left.\begin{array}{l} x+y=13 \\ xy=36 \end{array}\right\}$

(ii) $\left.\begin{array}{l} x+y=13 \\ xy=40 \end{array}\right\}$

5. Solve the equations geometrically (i) $\left.\begin{array}{l} x-y=6 \\ xy=40 \end{array}\right\}$

(ii) $\left.\begin{array}{l} x-y=6 \\ xy=72 \end{array}\right\}$

6. Extract geometrically the square root of the following numbers (i) 7 ; (ii) 8 ; (iii) 9.

7. Solve the following equations geometrically (i) $x^2-3x+2=0$; (ii) $x^2+2x-4=0$.

8. Draw a line 2·5″ long and divide it internally so that the rectangle contained by the segments contains 1 sq. inch area.

9. Draw an equilateral triangle on a base 3″ long. Construct an equivalent square and show that its sides are each 1·97″ long.

11. On a given straight line AB construct a rectangle equal to a given square.

12. On a given straight line AB construct a rectangle equal to a given rectangle.

13. Show that if a rectangle be equal to a square, the perimeter of the square is less than that of the rectangle.

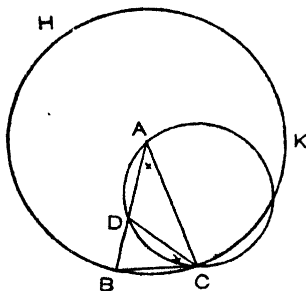
CHAPTER VII

CONSTRUCTION OF REGULAR PENTAGON

Lemma.

PROBLEM 47.

To construct an isosceles triangle having each of the angles at the base double of the vertical angle.



Construction. Take any str. line AB and divide it at D so that AB.BD $=$ AD^2.

With A as centre and AB as radius describe the ⊙HBK.

In this ⊙ place the chord BC $=$ AD.

Join AC.

Then ABC is the △ reqd.

Proof. Join CD, and circumscribe a ⊙ about the △ADC.

Since BC^2 $=$ AD^2, ∴ BC^2 $=$ AB.BD.

∴ BC touches the ⊙ ADC at C. *Th. 60.*

Hence, the ∠BCD $=$ the ∠DAC, in the alt. segment.

Hence, the ∠BDC = the ∠BCA,
because each of them = the sum of the ∠s DAC, ACD.

But the ∠ABC = the ∠BCA; (∵ AC = AB),
∴ the ∠ABC = the ∠BDC.

Hence, DC = BC = DA;
and ∴ the ∠DAC = the ∠DCA.

Hence, DC is the bisector of the ∠BCA.
and ∴ the ∠BCA = twice the ∠BCD
= twice the ∠BAC.
Hence the ∠CBA also = twice the ∠BAC.

Thus, ABC is the required △.

Q. E. F.

Analysis: *Assume* ABC to be a △ such that each of its ∠s B and C is double the ∠A.

Let CD bisect the ∠C. Then the ∠BCD = $\frac{1}{2}$ the ∠BCA = the ∠BAC. Hence BC must touch the circum-⊙ of the △ADC at C. *Th. 48, Cor.*

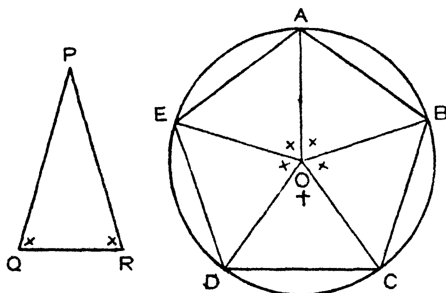
∴ BC^2 = AB.BD. The ∠DAC = the ∠DCA, (because each of them is $\frac{1}{2}$ the ∠BCA); ∴ DC = DA; also since the ∠DBC = the sum of the ∠s DAC, DCA = twice the ∠DAC = the ∠ABC; ∴ BC = DC. Hence BC = DA, and ∴ AB.BD = AD^2; which determines the position of D. Then the position of C also is determined, because the distance of C from A = AB, and its distance from B = DA.

Obs. Evidently 5 times the ∠A = two rt. ∠s. Hence the ∠A = one-fifth of two rt. ∠s, and ∴ each of the ∠s ABC, ACB = two-fifths of two rt. ∠s = *one-fifth of four rt. ∠s.*

Regular Pentagon

PROBLEM 48.

*To construct a regular figure of five sides (i) **in** or (ii) **about** a given circle.*



Let the ⊙ with centre O be the given ⊙.

It is required to construct an equilateral and equiangular pentagon (i) in or (ii) about the circle (O).

Construction. Construct an isosceles Δ PQR having each of the ∠ˢ at Q and R double of the ∠ at P. *Prob. 47.*

Draw any radius OA, and then draw the radius OB making the ∠AOB = the ∠Q.

Now, successively draw the radii OC, OD, OE making the ∠ˢ BOC, COD, DOE each = the ∠Q.

(i) Join AB, BC, CD, DE, EA. Then ABCDE is a regular pentagon inscribed in the given ⊙.

Proof. Each of the ∠ˢ at Q, R is one-fifth of four rt. ∠ˢ. *Prob. 47, Obs.*

Hence each of the ∠ˢ AOB, BOC, COD, DOE is one-fifth of four rt. ∠ˢ.

∴ the ∠EOA, which together with these four ∠ˢ makes up the four rt. ∠ˢ at O, must also be = one-fifth of four rt. ∠ˢ.

Thus the five ∠ˢ at O are equal;

∴ the five chords AB, BC, CD, DE, EA are equal.

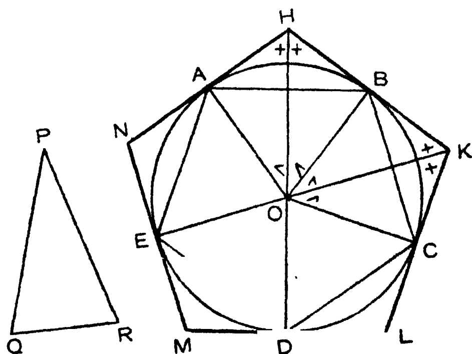
Hence, the pentagon ABCDE is equilateral.

Again, the arc AEDC = the arc BAED;
(∵ each of them = 3 times the arc AB),
∴ the ∠ABC = the ∠BCD. *Th. 43, Cor. 5.*

Similarly, every two consecutive ∠ˢ of the pentagon are equal.

Hence the pentagon ABCDE is both equilateral and equiangular, and it is also *inscribed* in the given ⊙. Q. E. F.

(ii) Draw tangents to the ⊙ at A, B, C, D, E forming the pentagon HKLMN. Then HKLMN is a regular pentagon circumscribed about the ⊙.



Proof. Join OH, OK.

It may be proved, as before, that the five radii OA, OB, OC, OD, OE divide the four rt. ∠ˢ at O into five equal parts.

Also the ∠ˢ at A, B, C, D, E are rt. ∠ˢ. *Th. 45.*

Now, since the tangents AH, BH meet at H,

$\therefore$ AH = BH,
the $\angle$AOH = the $\angle$BOH,
and the $\angle$OHA = the $\angle$OHB. } *Th. 48.*

Hence, the $\angle$BOH = $\frac{1}{2}$ the $\angle$AOB,
and the $\angle$OHB = $\frac{1}{2}$ the $\angle$AHB.

Similarly, the $\angle$BOK = $\frac{1}{2}$ the $\angle$BOC,
and the $\angle$OKB = $\frac{1}{2}$ the $\angle$BKC.

In the Δ^s OBH, OBK, we have

the $\angle$OBH = the $\angle$OBK; (*being rt.* $\angle^s$);

the $\angle$BOH = the $\angle$BOK, (*halves of the equal* $\angle^s$ AOB, BOC)

and the side OB common;

$\therefore$ the two Δ^s are congruent.

Hence BH = BK........................(α)
and the $\angle$OHB = the $\angle$OKB..............(β)

From (α), HK = 2BH; and in the same way it may be proved that NH = 2AH.

$\therefore$ NH = HK, ($\because$ AH = BH);

similarly HK = KL, KL = LM, LM = MN, and MN = NH.

Hence the pentagon HKLMN is equilateral.

Again from (β), $\angle^s$ AHB and BKC are equal. *i. e.* $\angle$H = $\angle$K.

Similarly, $\angle$K = $\angle$L = $\angle$M = $\angle$N.

$\therefore$ the pentagon is also equiangular.

Hence, it is regular. Q. E. F.

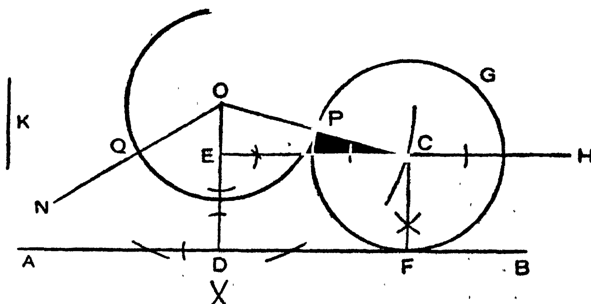
CHAPTER VIII

MISCELLANEOUS CONSTRUCTIONS

I. On Circles.

PROBLEM 49.

1. *To construct a circle, with a given radius touching a given circle and a given straight line.*



Let the ⊙ with centre O be the given ⊙, AB the given str. line, and K the given radius.

It is required to describe a circle which shall touch the circle (O) *as well as the straight line* AB, *and have a radius* = K.

Construction. Draw OD ⊥ to AB, and from DO cut off DE = K; through E draw EH ∥ to AB.

Take a radius OQ of the given ⊙ and produce it to N, making QN = K. Then ON is the sum of the radii of the given and the required ⊙s.

With O as centre and ON as radius describe an arc cutting EH at C.

Join OC cutting the given ⊙ at P, and draw CF ⊥ to AB.

Then the ⊙ described with C as centre and CP as radius will be the reqd. ⊙.

Proof. Since OC = ON, of which OP = OQ.

∴ CP = NQ = K;

also, since DC is a rectangle, by construction,

∴ CF = ED = K.

Hence CP = CF; and ∴ the ⊙ described with C as centre and CP as radius will also pass through F.

Now, since the two ⊙ˢ meet at P which is a pt. on the line joining the centres.

∴ the ⊙ˢ touch each other at P. *Th. 47, Note 2.*

Also, since AB is ⊥ to the radius CF at F.

∴ AB touches the ⊙ at F.

Th. 45, Cor. 2.

Thus, the ⊙ PFG touches each of the given ⊙ and the given str. line, and has its radius = K;

∴ this is the required ⊙.

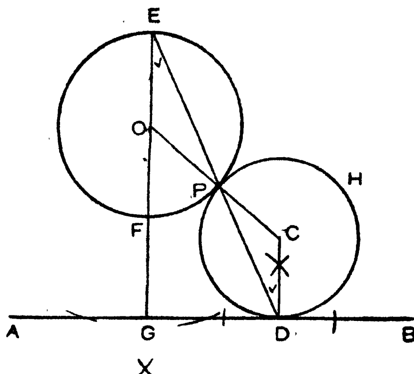
Q. E. F.

NOTE 1. The circle described with O as centre and ON as radius will also cut HE produced at some pt. With that point as centre another ⊙ may be constructed satisfying the conditions of the problem. This is left as an exercise for the student.

NOTE 2. If the reqd. ⊙ is *assumed* to be drawn, it is easy to see that the distance of its centre from O is equal to the sum of the radii, and the distance from AB = K. Hence follows the construction.

PROBLEM 50.

To construct a circle touching a given circle and also touching a given straight line at a given point.



Let AB be the given str. line and D the given pt. in it, and let the ⊙ with centre O be the given ⊙.

It is required to describe a circle touching the circle (O) *and also touching* AB *at* D.

Construction. Draw OG ⊥ to AB cutting the ⊙ at F, and produce GO to meet the ⊙ in E.

(i) Join ED cutting the ⊙ at P; join OP.

Through D draw a perpendicular to AB, meeting OP produced in C.

Then the ⊙ described with C as centre and CP as radius will be the reqd. ⊙.

Proof. Since EG and CD are both ⊥ to AB,

∴ they are parallel;

∴ the ∠OEP = the ∠CDP; (*alt.* ∠*s*)

also, the ∠OPE = the ∠CPD. (*vertically opp.* ∠*s*)

But the ∠OEP = the ∠OPE; (∵ OP = OE)

∴ the ∠CDP = the ∠CPD;

and ∴ CP = CD.

Hence the ⊙ described with C as centre and CP as radius will also pass through D.

Again, since the two ⊙s meet at P which is on the line joining the centres, they touch each other at P.

Th. 47, Note 2.

Also, since AB is ⊥ to the radius CD at D,

∴ AB touches the ⊙ at D. *Th. 45, Cor. 2.*

Thus, the ⊙ PDH touches the given ⊙, and also touches the given str. line at the given pt. D;

∴ this is the reqd. ⊙. Q. E. F.

(ii) Join DF and produce it to meet the ⊙ at P. Join PO.

Through D draw a ⊥ to AB, meeting PO produced in C.

Then the ⊙ described with C as centre and CP as radius will be the required ⊙.

[The proof is left as an exercise for the student.]

NOTE 1. Let the required circle be *assumed* to be drawn, as in the first diagram; let C be its centre and P the pt. where it touches the given ⊙. Then OC passes through P, and CD is ⊥ to AB. Join DP and produce it to meet the given ⊙ in E; join EO. Now, it is easily seen that the ∠CDE = the ∠OED; ∴ EO is ∥ to CD, and ∴ EO produced cuts AB at rt. ∠s. We thus find that the pt. of contact of the two ⊙s lies on the line DE, where OE is that radius of the given ⊙ which, when produced through O, cuts AB at rt. ∠s. The position of P being known, that of C is found at once, for it lies on OP produced and also on the ⊥ to AB through D.

NOTE 2. Let the reqd. circle be *assumed* to be drawn, as in the second diagram; let C be its centre and P the pt. where it touches the given ⊙. Then CO produced passes through P, and CD is ⊥ to AB. Join DP cutting the given ⊙ at F; join OF. Now, it is easily

seen that the ∠CDP = the ∠OFP ; ∴ OF is ∥ to CD, and ∴ OF produced cuts AB at rt. ∠s. We thus find that the pt. of contact of the two ⊙s lies on the line DF produced, where OF is that radius of the given ⊙ which, when produced through F, cuts AB at rt. ∠s. The position of P being known, that of C is found at once, for it lies on PO produced and also on the ⊥ to AB through D.

PROBLEM 51.

To construct a circle passing through two given points and touching a given circle.

Let A, B be the two given pts. and let the ⊙ with centre O be the given ⊙.

It is required to describe a circle passing through A *and* B, *and also touching the circle* (O).

Join AB and draw FC, the ⊥ bisector of AB.

Then, FC may pass through O or may not.

(i) Let FC not pass through O.

Construction. With any pt. C, on FC, as centre, and CA or CB as radius, describe a ⊙ cutting the given ⊙ at D and E. Join DE and produce it to meet AB produced at H.

Draw HP to touch the given ⊙ at P. *Problem 32.*

Construct a ⊙ through the pts. A, B, P ; then this will be the reqd. ⊙.

Proof. Since the chords AB, DE of the ⊙ABD intersect in H,

∴ HA.HB = HD.HE. *Th. 57.*

Again, since HP touches the given ⊙ at P and HD cuts it in D and E,

∴ HP^2 = HD.HE. *Th. 59.*

Hence, HA.HB = HP^2,

and ∴ HP touches the ⊙ ABP at P. *Th. 60.*

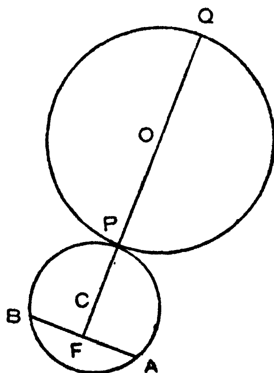
Now, the ⊙ ABP and the given ⊙ have a common tangent at P,

∴ the two ⊙s touch each other at P. *Th. 47, Cor. 3.*

Hence the ⊙ABP is the reqd. ⊙.

[Similarly, if the other tangent HQ to the given ⊙ be drawn, it may be proved as above that the ⊙ through A, B, Q is another ⊙ satisfying the given conditions.]

(ii) Let FC pass through O.



Construction. Let FO cut the given ⊙ at P.

Describe a ⊙ passing through the pts. A, B, P; then this will be the reqd. ⊙.

Proof. Since FP is the ⊥ bisector of AB,

∴ the centre of the ⊙ ABP lies in FP.

Hence, P is a pt. on the line joining the centres of the two ⊙ˢ.

and since the ⊙ˢ meet at P ;

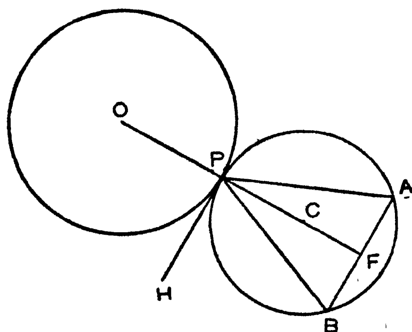
∴ they touch each other at P. *Th. 47, Note 2.*

Hence the ⊙ APB is the reqd. ⊙. Q. E. F.

[If FO be produced to meet the given ⊙ in Q, it may be proved as above, that the ⊙ through ABQ is another ⊙ satisfying the given conditions.]

Analysis : *Assume* the required ⊙ to be drawn, touching the given ⊙ at P. Then the two ⊙ˢ have a common tangent at P ; let this tangent meet AB produced in H. ∴ $HP^2 = HA.HB$, (Th. 59). Through H let any secant be drawn cutting the given ⊙ in E and D ; then $HD.HE = HP^2$ (Th. 59). Hence $HA.HB = HD.HE$, which shows that the pts. A, B, E, D are concyclic (Th. 58). Hence, if a ⊙ be constructed passing through A, B and also cutting the given ⊙ in two pts. D, E, then the pt. in which DE intersects AB is the pt. in which the common tangent to the given and the reqd. ⊙ˢ intersects AB ; whence the construction follows.

If the common tangent at P *be* ∥ *to* AB, *it will not meet* AB *produced.*

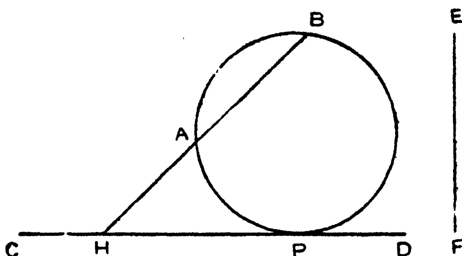


Join PA, PB. Since PH is ∥ to AB, ∴ the ∠HPB = the ∠PBA; also the ∠HPB = the ∠PAB, in the alt. segment.

Hence the $\angle$PBA = the $\angle$PAB, and $\therefore$ PA = PB. Hence P is on the $\perp$ bisector of AB. The centre C of the $\odot$ APB also lies on the $\perp$ bisector of AB. Hence both C and P lie on the $\perp$ bisector of AB, and $\therefore$ PC produced bisects AB at rt. $\angle$s. But CP produced also passes through O. Hence OP produced bisects AB at rt. $\angle$s. In this case therefore the $\perp$ bisector of AB passes through O and cuts the given $\odot$ in the pt. at which the reqd. $\odot$ touches it; whence follows the construction.

PROBLEM 52.

To construct a circle passing through two given points and touching a given straight line which is not parallel to that joining the given points.



Let A, B be the two given points and CD be the given str. line intersecting the str. line AB, when produced, at H.

It is required to draw a circle through A, B *and touching the str. line* CD.

Construction. Find a str. line EF the square on which is equal in area to the rect. BH.HA.

From HD cut off HP = EF.

Then the $\odot$ passing through the pts. A, B, P will be the reqd. $\odot$.

Proof. Since $HP^2 = HA.HB$,

$\therefore$ HP touches the $\odot$ at P. *Th. 60.*

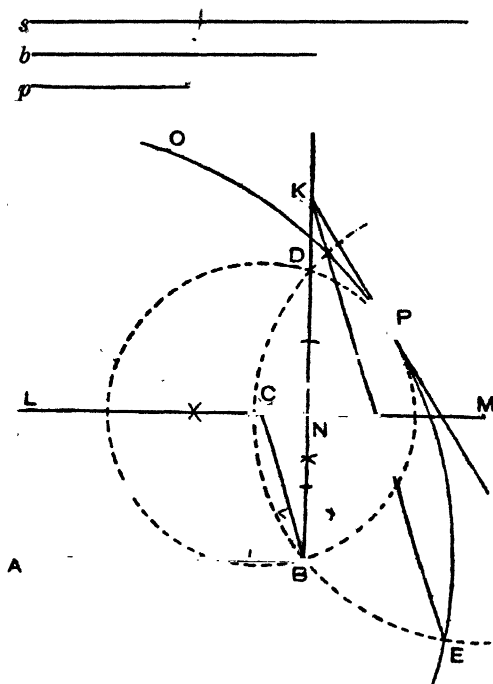
Hence ABP is the reqd. $\odot$. Q. E. F.

NOTE. HP may as well be cut off from HC produced. Thus there are two $\odot$s satisfying the given conditions.

II. On Triangles.

PROBLEM 53.

To construct a triangle, having given the base, the altitude and the sum of the other two sides.



Let s, b, p (fig. above) denote respectively the lengths of the sum of the two sides, the base and the altitude of a triangle.

It is required to construct the triangle.

Construction. Draw any str. line AB $= b$. With centre A and radius s, draw the ⊙ OPE ; draw a str line LM par^l to AB and at a distance 'p' from it. From B, draw BN perp. on LM and produce it to D, making ND $=$ BN. Draw a ⊙ DPB through B and D and touching the ⊙ OPE. If P be the point of contact, join AP and let it cut LM at C. Then, ACB is the reqd. triangle.

Proof. Because the two ⊙' OPE, DPB touch at P, internally, their centres are collinear with P, the point of contact.

Therefore, the centre of the ⊙ DPB must lie on AP.

Also, its centre must lie on the perp. bisector, LM, of the chord DB.

∴ the centre of the ⊙DPB is the pt. C, at which LM and AP intersect.

Hence, because AC $+$ CB $=$ AC $+$ CP $=$ AP $= s$, and the perp. distance of C from AB $= p$ (for, C lies on LM), the Δ ACB is the reqd. triangle.

Obs. It may be easily verified that in the above construction if the given altitude be such that the point D (where BD $= 2p$) is outside the ⊙OPE, the point K will be inside the same ⊙ ; hence, the ⊙DPB which passes through D, B and P (the pt. of contact of the tangent to the ⊙OPE from K) cannot be drawn. Hence the construction fails.

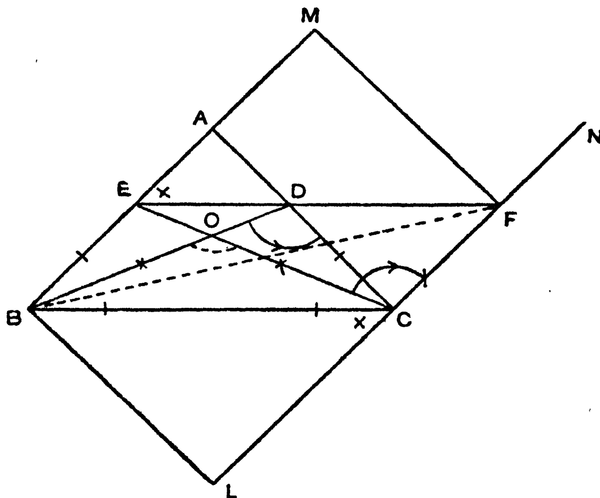
Exercise : Show that there are two Δs satisfying the given conditions.

CHAPTER IX

MICELLANEOUS PROPOSITIONS

I. On Rectilineal Figures and Circles.

1. *If the bisectors of the base angles of a triangle be equal then the triangle is isosceles.* (*B. C. S. Exam.*)



Let ABC be a triangle and BD, CE be the bisectors of the base ∠ˢ B and C.

If BD = CE, *it is reqd. to prove that the* △ ABC *is isosceles.*

At C draw CN, making, the ∠ECN = the ∠BDC ; and cut off from CN, CF = CD. From B and F, draw BL, FM perpˢ on FC and BA (produced if necessary) respectively.

Join EF, BF.

Proof. In the Δ^s BDC, ECF,

∵ BD = CE *Hyp.*

CD = CF *Cons.*

and the ∠BDC = the ∠ECF *Cons.*

∴ the Δ^s are congruent.

∴ (i) EF = BC; (ii) the ∠DBC = the ∠FEC.

Now, from the ΔBOE,

the ext. ∠BOC = the ∠OBE + the ∠OEB

= the ∠OBC + the ∠OEB

[∵ BD *bisects* the ∠B]

= the ∠CED + the ∠OEB, *from* (*ii*) *above*

= the ∠BED.

Again from the ΔODC,

the ext. ∠BOC = the ∠ODC + the ∠OCD

= the ∠OCF + the ∠OCB

= the ∠BCF.

∴ the ∠BED = the ∠BCF.

∴ the ∠FEM = supplement of the ∠BED

= *supplement* of the ∠BCF = the ∠BCL.

Hence, in the rt. angled Δ^s FEM, BCL

∵ BC = EF, *from* (*i*)

and the ∠FEM = the ∠BCL

∴ the Δ^s are congruent.

∴ (iii) EM = CL; (iv) FM = BL.

Again, from the rt. ∠^{ed} Δ^s BFM, BFL,

∴ FM = BL

and the hypotenuse BF is common

∴ the Δ^s are congruent,

∴ (v) BM = FL.

∴ from (iii) and (v),

BM − EM = FL − CL

i.e., BE = CF.

But CF = CD, by construction.

∴ BE = CD.

Hence, in the two Δ^s BEC, BDC

$\because$ BD = CE *Hyp.*

BE = CD

and BC is common ;

$\therefore$ the Δ^s are congruent.

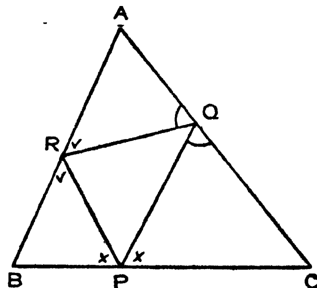
$\therefore$ the $\angle$EBC = the $\angle$DCB

i.e. the $\angle$ABC = the $\angle$ACB.

i.e. the ΔABC is isosceles.

2. *Of all triangles inscribed in a given acute-angled triangle it is the pedal triangle alone which has two of its sides equally inclined to that side of the given triangle on which they meet.*

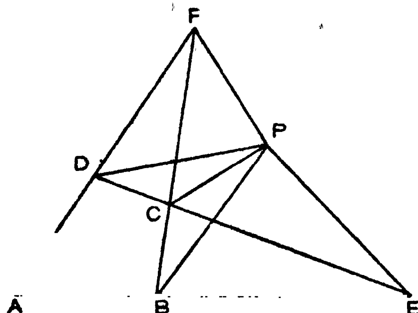
Let the ΔPQR inscribed in the acute-angled ΔABC be such that its sides RP, QP are equally inclined to BC, the sides PQ, QR are equally inclined to CA, and the sides QR, RP are equally inclined to AB, as shown in the accompanying diagram.



It is required to show that the ΔPQR *is the pedal triangle of the* ΔABC.

Proof. Now, the six $\angle^s$ marked at P, Q, R, together with the $\angle^s$ at A, B, C make up *six* rt. $\angle^s$; hence, those *six* $\angle^s$ are together = 4 rt. $\angle^s$. Hence the *three* $\angle^s$ AQR, ARQ and QPC (one being taken from each pair) are together = 2 rt. $\angle^s$; and $\therefore$ the $\angle$QPC = the $\angle$A, which shows that the quadl. ABPQ is cyclic. Similarly, each of the quadls. BCQR and CARP is cyclic. Hence, if AP, BQ and CR be joined, it is easy to see that the $\angle$APQ = the $\angle$ABQ = the $\angle$RCQ = the $\angle$APR ; whence the $\angle$APC = the $\angle$APB ; and $\therefore$ AP is $\perp$ to BC. Similarly BQ is $\perp$ to CA and CR is $\perp$ to AB. Hence the ΔPQR is no other than the pedal triangle. Thus, of all the Δ^s that can be inscribed in the ΔABC, it is only the pedal Δ that has the above property. Q. E. D.

3. *The circum-circles of the four triangles formed by four given straight lines, of which no two are parallel, have one point common to them all.*



Let the four str. lines AB, DC, AD, BC form the four Δ' ABF, DCF, ADE, BCE. C is clearly one pt. of intersection of the ⊙' DCF, BCE ; let their second pt. of intersection be P.

It is required to prove that P *is a pt. on the* ⊙ABF *and also on the* ⊙ADE.

Proof. Join CP, BP, DP.

The quadl. BCPE being cyclic, the ∠PBE = the ∠PCE ; *Th. 37.*

and the quadl. DCPF being cyclic. the ∠PCE = the ∠PFD. *Th. 39, Cor. 1.*

Hence, the ∠PBE = the ∠PFA,

and ∴ the quadl. ABPF is cyclic. *Th. 40, Cor.*

Thus, P is a pt. on the ⊙ ABF.

In the same way, it can be shown that P is a pt. on the ⊙ ADE.

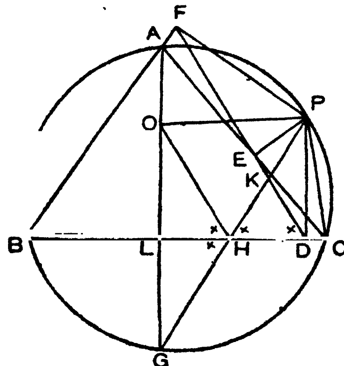
COR. 1. *If the four given straight lines are such that the quadl.* ABCD *is cyclic, then the three pts.* F, P, E *are collinear.*

For, in that case, the ∠CDA = the ∠CBE ; ∴ the ∠' FPC and EPC are together = the ∠' CBE and EPC = two rt. ∠'. Hence, FP and PE are in the same str. line.

COR. 2. *If from* P *the* ⊥' PH, PK, PL, PM *be drawn to the lines* ABE, DCE, BCF, ADF *respectively, then the four pts.* H, K, L, M *are collinear.*

Since P is a pt. on the circum-⊙ of the △BCE, and PH, PK, PL are the perp^s from P on the sides of the triangle. ∴ the three pts. H, K, L lie in one str. line. (Simson's Line Page 236). Similarly, the three pts. K, L, M are in one str. line. Thus, the str. line that passes through K and L also passes through H and M; which shows that all the four pts. lie in one str. line.

4. *The line joining the ortho-centre* O *of a triangle* ABC *to any point* P *on the circum-circle is bisected by the pedal line of* P *with respect to the triangle.*



In the above diagram, let DEF be the pedal line of the pt. P.

It is required to prove that OP *is bisected by the line* DEF.

Proof. Produce AO to meet BC in L and the circum-⊙ in G. Join PG, cutting BC at H, and the pedal line at K.

Join OH, PC. The line AOL is ⊥ to BC.

Thus, if BO, BG be joined,

the ∠GBC = the ∠GAC = the complement of the ∠ACB
= the ∠CBO.

∴ from the △^s LBO, LBG, in which the ∠BLO
= the ∠BLG, the ∠GBL = the ∠LBO and BL is common, we have, OL = LG.

Hence the △^s OLH, GLH are congruent;

and ∴ the ∠LHG = the ∠LHO.

Now, the quadl. PEDC being cyclic,

the ∠EDH = the ∠EPC

= the complement of the ∠PCA

= the complement of the ∠PGA

= the ∠LHG

= the ∠LHO ;

∴ DF is ∥ to HO.

Again, the ∠KDH = the ∠LHG *Proved above.*

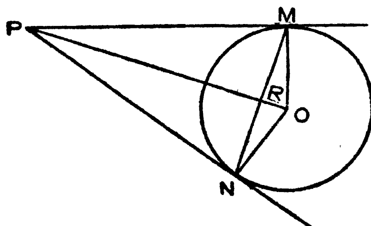
= the ∠KHD}
∴ the ∠KDP = the ∠KPD.}

Hence, KH = KD = KP.

Thus, K is the mid-pt. of the side PH of the △OPH, and the line DKF is ∥ to the side HO ;

∴ the side OP is bisected by the line DEF. Q. E. D.

5. P *is any point outside a circle of which the centre is* O ; PM, PN *are tangents to the circle, and* MN *the chord of contact. If* OP *intersects* MN *at* R, *then* OR.OP = OM^2.



Proof. Since the ∠ˢ OMP, ONP are supplementary, each of them being a rt. ∠,

∴ the quadl. OMPN is cyclic.

Hence, the ∠OMN = the ∠OPN *Th.* 37.

= the ∠OPM. *Th.* 46, *Cor.*

Hence, OM touches the ⊙ circumscribed about the △PMR at M ; *Th.* 48, *Cor.*

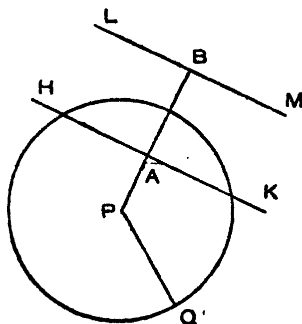
∴ OR.OP = OM^2. *Th.* 59.

Obs. PM being=PN, the ⊥ bisector of MN passes through P; for a similar reason, the ⊥ bisector of MN passes through O. Hence OP is the ⊥ bisector of MN. Thus, *the chord of contact of tangents drawn to a circle from an external point is bisected perpendicularly by the line joining that point to the centre.*

6. Definition: If two points are so related to a circle that they lie in a straight line passing through the centre, and the rectangle contained by their distances from the centre is equal to the square on the radius, then each of these points is said to be the **inverse** of the other, with respect to the given circle.

Thus, in the preceding diagram, the points R and P are a pair of *inverse points* with respect to the circle whose centre is O and radius=OM.

Obs. If A and B are a pair of *inverse points* with respect to a circle whose centre is P and radius= PQ (that is, if P, A, B are collinear and $PA.PB=PQ^2$), and if through A a straight line HK be drawn ⊥ to the line PAB, then the point B is said to be the **pole** of the line HK, and the line HK, is said to be the **polar** of the point B. Also, if LM be drawn through B perpendicular to the line PAB, then A is the *pole* of the line LM, and the line LM is the *polar* of the point A. Thus, in the preceding diagram, the point P is the *pole* of the line MN and MN is the *polar* of the point P.

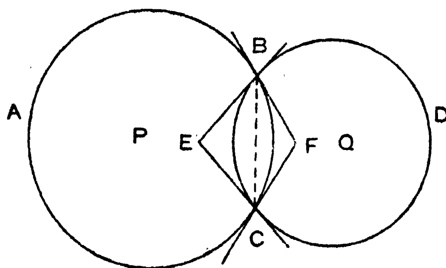


II. Orthogonal Circles.

7. Definition: Two circles are said to be **orthogonal,** or to intersect **orthogonally,** when the two tangents at either point of intersection are perpendicular to each other.

In the diagram of page 305, M is a pt. of intersection of the given ⊙ and the ⊙ about the △PMR; also MP and MO, which are the tangents at M to those two ⊙s, respectively, are ⊥ to each other. Hence the given ⊙ and the ⊙ about the △PMR intersect *orthogonally.* Similarly, the given ⊙ and the ⊙ about the △PNR intersect *orthogonally.*

1. *The angle between the tangents to any two intersecting circles at one of their points of intersection is equal to the angle between the tangents at the other point.*



Let ABC, DBC be any two ⊙ˢ, intersecting each other at B and C; and let the tangents at B and C to the two ⊙ˢ intersect at E and F respectively, as is shown in the diagram.

It is required to prove that the ∠EBF=*the* ∠ECF.

Join BC.

Proof. Since the tangents at B and C to the ⊙BDC intersect at E,

EB=EC.

∴ the ∠EBC=the ∠ECB.

Similarly, FB=FC.

∴ the ∠FBC=the ∠FCB

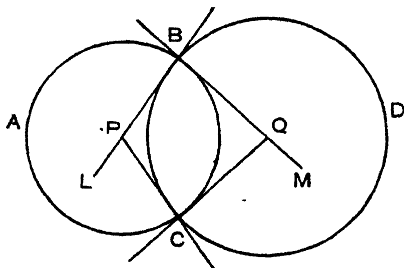
∴ the ∠EBC+the ∠FBC=the ∠ECB+the ∠FCB

i.e. the ∠EBF=the ∠ECF.

COR. 1. EF *bisects the angles* BEC, BFC.

COR. 2. *If the* ⊙ˢ ABC, DBC *be orthogonal to each other, the four points* E, B, F, C *are concyclic.*

2. *If two circles intersect orthogonally, the tangents at their points of intersection to either of them pass through the centre of the other.*



Let ABC, DBC be any two ⊙ˢ, intersecting orthogonally at B, C and let P and Q be their centres, respectively.

It is required to prove that (i) the tangents at B *and* C *to the* ⊙DCB *pass through* P ;

and (ii) the tangents at B, C *to the* ⊙ABC *pass through* Q.

Proof. Because BL is a tangent at B to the ⊙DBC, and BM is prep. to BL through the pt. of contact,

∴ BM must pass through the centre Q of the ⊙DBC.

But BM is a tangent to the ⊙ABC ;

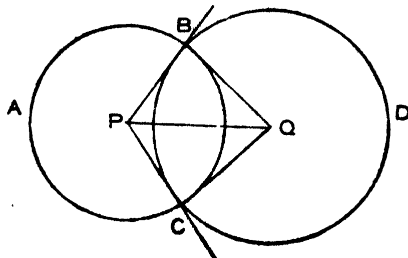
∴ the tangent at B to the ⊙ABC passes through Q, the centre of the ⊙DBC.

Similarly, the tangent BL to the ⊙DBC passes through P, the ceutre of the ⊙ABC.

In like manner, it can be shown that the tangents at C to either of the two ⊙ˢ pass through the centre of the other.

COR. *The angles subtended by the common chord of any two orthogonal circles at their centres are supplementary.*

3. *The square on the distance between the centres of any two orthogonal circles is equal to the sum of the squares on their radii.*



Let P, Q be the centres of two orthogonal circles ABC, DBC, which intersect at B and C.

It is required to prove that sq. on PQ = *sum of the squares on the radii of the two* ⊙ˢ.

Join PQ, PB, QB.

Proof. Since the ⊙ˢ intersect orthogonally,

PB, QB are tangents to the two ⊙ˢ, one to each; and they include a rt. ∠ between them. *Prop. 2, p, 308.*

∴ the △PBQ is a rt. ∠ᵉᵈ △, of which PQ is the hypotenuse.

∴ $PQ^2 = PB^2 + QB^2$ *Pythagoras' Theorem.*

= sum of the squares on the radii of the two ⊙ˢ,

because, PB and QB are the radii of the two ⊙ˢ.

Obs. If d be the distance between P and Q, and r_1, r_2 be the radii of the two circles, the above relation is :—

$$d^2 = r_1^2 + r_2^2 ;$$

$$\text{or,} \quad 2d^2 = 2(r_1^2 + r_2^2)$$

$$= (r_1 + r_2)^2 + (r_1 - r_2)^2$$

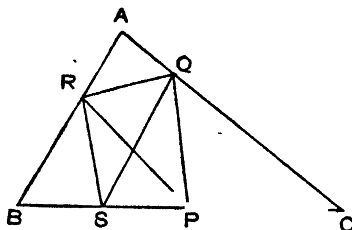
$$\text{i.e.} \quad > (r_1 + r_2)^2 ;$$

$$\text{whence, } r_1 + r_2 < \sqrt{2}.d,$$

i.e. the sum of the radii of two orthogonal ⊙ˢ is less than $\sqrt{2}$-times the distance between their centres.

III. Maxima-Minima.

1. *Of all triangles that can be inscribed in an acute-angled triangle, the pedal triangle is that which has the least perimeter.*

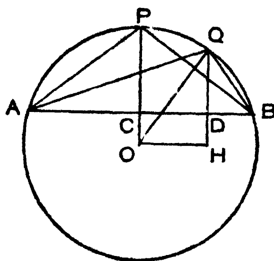


Let ABC be an acute-angled Δ.

It is required to show that the pedal triangle of the ΔABC *has the least perimeter.*

Proof. Now, of all the Δ' that can be inscribed in the ΔABC, *either* the pedal Δ, *or* some one of the *others,* must have the least perimeter. Let PQR be *any* of the inscribed Δ' *other than* the pedal Δ, as in the above diagram. Since PQR is not the pedal Δ, it must have *at least* one pair of sides that are *not* equally inclined to that side of the ΔABC on which they meet ; let RP, PQ be that pair. Suppose then that S is the pt. in BC such that RS, SQ are equally inclined to it ; then RS+SQ is < RP+PQ, and ∴ the perimeter of the ΔRSQ is < that of the ΔRPQ. The ΔPQR is ∴ *not* one having the least perimeter. Hence, the Δ of minimum perimeter is clearly *no other* than the pedal Δ ; or, which is the same as, it is the pedal Δ that has the minimum perimeter.'

2. *Of all triangles standing on the same base and having the same vertical angle, the isosceles is that which has the greatest area and the greatest perimeter.*



Let the isos. Δ APB stand on the base AB; and let AQB be *any other* Δ on the *same* base, having the vertical $\angle$ AQB= the $\angle$ APB.

(*i*) *It is required to prove that the area of the* Δ APB *is greater than the area of the* Δ AQB.

Proof, Since the $\angle$ APB=the $\angle$ AQB,

$\therefore$ the four pts. A, P, Q, B are concylic. *Th. 38.*

Let O be the centre of the $\odot$ passing through these four points, and draw OC, QD $\perp$ to AB.

Then OC is the $\perp$ bisector of AB; *Th. 33.*

and since PA=PB, $\therefore$ OC produced must pass through P.

Draw OH $\perp$ to QD produced; join OQ.

Now, in the rt. $\angle^{ed}$ Δ OQH, the hypotenuse OQ>QH.

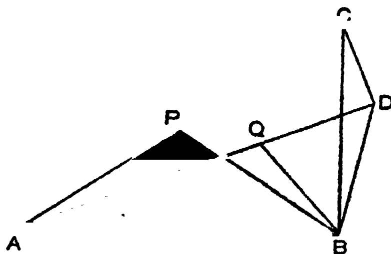
Hence OP>QH,
and, CH being a rectangle, CO=DH;
$\therefore$ PC>QD.

Now, the area of the Δ APB=$\frac{1}{2}$ the rect. AB.PC;
and that of the Δ AQB=$\frac{1}{2}$ the rect. AB.QD.

Hence the area of the Δ APB> the area of the Δ AQB.

Q. E.

(*ii*) *It is required to prove that the perimeter of the* △ APB *is greater than the perimeter of the* △ AQB.



Proof. Produce AP to C making PC = PB, and AQ to D making QD = QB.

Join CB, DB, CD.

Since AP = PB = PC, ∴ the ⊙ described on AC as diameter will pass through B ;

∴ the ∠ABC is a rt ∠.

Now, since PC = PB, ∴ the ∠PBC = the ∠PCB,

and ∴ the ∠APB = double the ∠ACB ;
similarly, the ∠AQB = double the ∠ADB.

Hence, the ∠ACB = the ∠ADB ;
and ∴ the quadl. ACDB is cylic. *Th. 38.*

Hence, the ∠ADC = the ∠ABC
= a rt. ∠. *Th. 41.*

Now, in the rt. ∠[ed] △ADC,
the hypotenuse AC > the side AD ;

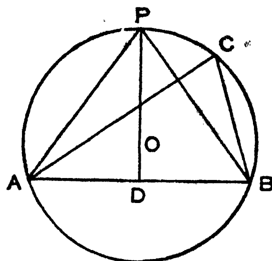
∴ AP + PB > AQ + QB.

Hence the perimeter of the △APB is > the perimeter of the △AQB.

Q. E. D.

m

3. *Of all triangles that can be inscribed in a gvien circle, that which has the greatest area, or the greatest perimeter, is equilateral.*



Let ABC be the given ⊙ of which the centre is O.

(*i*) *It is required to prove that of all* △ˢ *that can be inscribed in the* ⊙ABC, *that which has the greatest area is equilateral.*

Proof. The △ which has the greatest area must be *either* equilateral *or* non-equilateral.

Let ABC be *any* non-equilateral △ inscribed in the given ⊙.

It must then have *at least* one pair of unequal sides ; let AC, CB be that pair.

Draw OD ⊥ to AB, and produce DO to meet the ⊙ in P.

Join PA, PB.

Now, PD is the ⊥ bisector of AB.

∴ PA = PB.

Hence, of the two △ˢ APB, ACB, which stand on the same base AB and have equal vertical angles, the △ APB is isosceles.

Hence the area of the △ APB is greater than that of the △ ACB.

Thus, so long as the △ ACB has got a pair of *unequal* sides, another triangle of greater area can always be found.

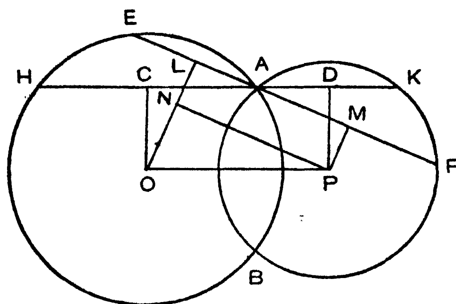
Hence, the △ of maximum area inscribed in the ⊙ *cannot* be non-equilateral ;

∴ it must be equilateral.

(*ii*) *To prove that of all* △ˢ *that can be inscribed in the* ⊙ABC, *that which has the greatest perimeter is equilateral.*

[The proof is similar to the above and is left as an exercise for the student.]

4. *Two circles whose centres are* O *and* P *cut each other at the points* A *and* B ; *of all the lines drawn through* A *or* B *and terminated by the two circumferences, the greatest is that which is parallel to* OP.



Let HK be the line through A ∥ to OP, and let EF be *any* other line through A, as in the above diagram.

It is required to prove that HK *is* > EF.

Proof. Draw OC, PD ⊥ to HK ; OL, PM ⊥ to EF, and PN ⊥ to OL.

Since OC is ⊥ to HA.

∴ HA = 2AC. *Th. 33.*

Similarly, AK = 2AD ;

∴ HK = 2CD.

In the same way, EF = 2LM.

Now CP and NM are evidently rectangles ;

∴ CD = OP,

and LM = NP.

But, in the rt. ∠^{ed} △ OPN,

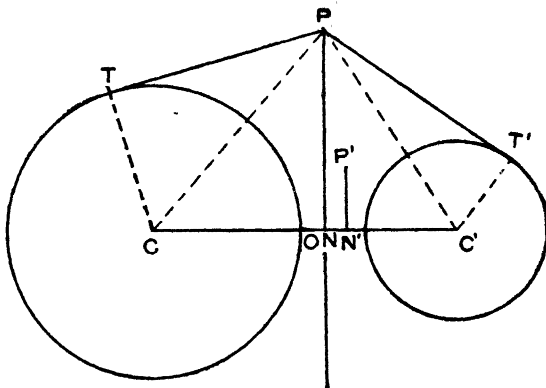
the hypotenuse OP > the side NP ;

∴ CD > LM, and ∴ HK > EF.

Q. E. D.

IV. Radical Axis.

1. *Find the locus of points from which tangents to two given non-intersecting circles ars equal.*



Let there be two non-intersecting circles whose centres are C and C′, and whose radii are r and r', respectively, as shown in the diagram.

Suppose P to be such a point that the tangents PT, PT′ drawn from P to the circles, are equal.

It is required to find the locus of P.

Form P, draw PN ⊥ to CC′. Join CT, CP, C′T′, C′P.

Bisect CC′ at O.

Proof. Because the ∠ˢ CTP, CNP are rt. ∠ˢ,

$\therefore \quad PT^2 = CP^2 - CT^2 = (CN^2 + PN^2) - r^2.$

Again, $\because$ the ∠ˢ C′T′P, C′NP are rt. ∠ˢ,

$\therefore \quad PT'^2 = C'P^2 - C'T'^2 = (C'N^2 + PN^2) - r'^2,$

$\therefore \quad CN^2 + PN^2 - r^2 = C'N^2 + PN^2 - r'^2,$

[$\because$ PT = PT′.]

i.e. $CN^2 - C'N^2 = r^2 - r'^2$, *taking away* PN^2 *from both sides.*

$\therefore \quad (CN + C'N)(CN - C'N) = r^2 - r'^2$ ....(1)

But $CN + C'N = CC'$; and $CN - C'N = (CO + ON) - (C'O - ON)$

$= 2.ON, \because CO = C'O.$

$\therefore$ (1) becomes $2CC'.ON = r^2 - r'^2$ ....(2)

Similarly, if P′ be another point such that tangents to the two ⊙ˢ from P′ are equal and if P′N′ be the perp. to CC′, it may be shown that

$$2CC'.ON' = r^2 - r'^2 \text{..........................(3)}$$

∴ from (2) and (3), ON = ON′, which is impossible, unless N and N′ coincide.

Thus, PN, PN′ are both perpˢ to CC′ through the same point, N, which is impossible, unless PN, P′N coincide *entirely*.

∴ P′ must lie on PN.

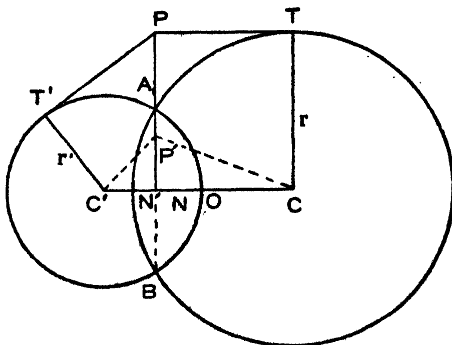
Hence, all points from which tangents to the two ⊙ˢ are equal, lie on the same str. line *viz*. the perp. through N to CC′, the line of centres of the two ⊙ˢ.

Hence, the required locus is a str. line perp. to the line of centres of the two circles.

8. Definition : The locus of points from which tangents to two given circles are equal is called the **radical axis** of the two circles.

It is clear from what has been shown in the last proposition that *the radical axis of two given circles is at right angles to the line of their centres.*

2. *The radical axis of two given intersecting circles is the same as their common chord.*



Let there be two intersecting circles whose common chord is AB; and of which the centres are C, C′ and radii are r, r' respectively.

It is required to show that the radical axis of the circles will coincide with AB.

Proof. Let P be any point outside the two given ⊙^{s} such that the tangents PT, PT′, drawn from P to the two ⊙^{s} are equal. If PN be the perp. from P to CC′, then, by the last proposition,

$$2CC'.ON = r^2 - r'^2, \text{ where O is the mid-pt. of } CC'.$$

Let the common chord AB meet CC′ at N′, so that AN′ is ⊥ to CC′. Suppose P′ to be any point in AB. Join CP′, C′P′.

Now, $CP'^2 - CA^2 = AP'.P'B = C'P'^2 - C'A^2$.

$\therefore \quad CP'^2 - C'P'^2 = CA^2 - C'A^2 = r^2 - r'^2$ ....(1)

But $\quad CP'^2 = CN'^2 + P'N'^2$, and $C'P'^2 = C'N'^2 + P'N'^2$

$\therefore \quad$ (1) becomes $CN'^2 - C'N'^2 = r^2 - r'^2$;

or, $\quad (CN' + C'N')(CN' - C'N') = r^2 - r'^2$

or, $\quad 2CC'.ON' = r^2 - r'^2$.

Hence, $2CC'.ON = 2CC'.ON'$, $(each = r^2 - r'^2)$.

$\therefore$ ON = ON′, which is impossible, unless N and N′ coincide.

$\therefore$ the perp. drawn to CC′ from the point P meets CC′ at the same point where the common chord meets it.

Also, since the common chord, also, is perp. to CC′, P must be a point on the common chord, AB, *produced*.

Similarly, all points from which tangents to the two circles are equal lie on the common chord *produced*.

Hence, the radical axis of two given circles is the same as their common chord.

Obs. Although *no* tangent can be drawn to any two intersecting circles from any point on their common chord, and as such it does not satisfy the condition of being a radical axis, *the entire straight line* obtained by producing the *common chord* bothways is *still* called **the radical axis** of the two circles.

3. *The radical axes of any three non-intersecting circles, taken in pairs, either meet in a point or are all parallel.*

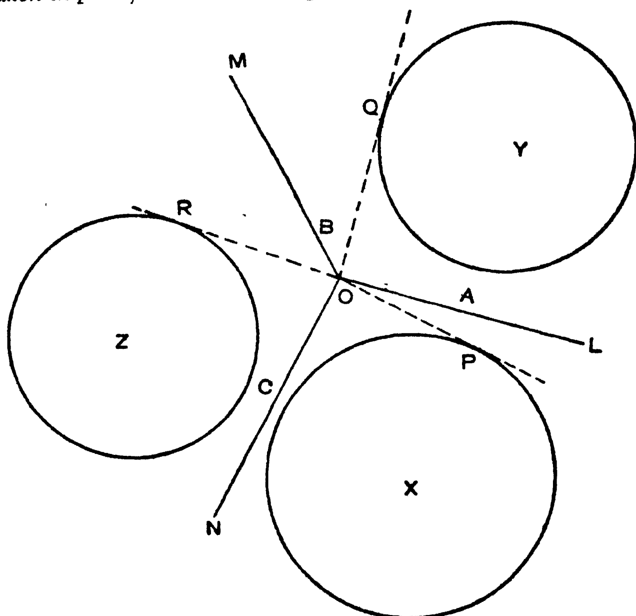


Fig. 1.

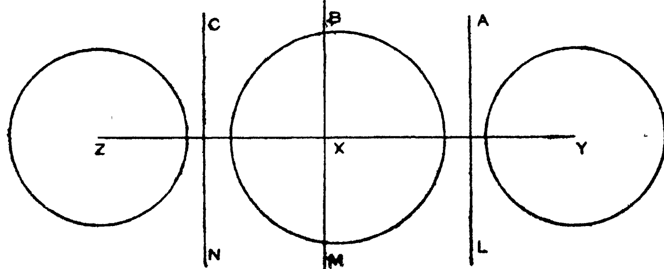


Fig. 2.

Let X, Y, Z be the centres of three non-intersecting circles;

and let the radical axis of the circles X, Y be AL, that of Y, Z be BM and that of Z, X be CN.

It is required to prove that (i) if two of the radical axes, say AL, BM, *meet at a point* O, *then the third one,* CN, *must pass through* O ;

or, (ii) if AL, BM *be parallel,* CN *is also par^l to them.*

Proof. (i) First suppose AL, BM meet at O. Draw OP, OQ, OR tangents to the circles X, Y, Z respectively.

Then, ∵ O is a pt. on AL, ∴ OP = OQ ;

also, ∵ O is a pt. on BM, ∴ OQ = OR.

∴ OP = OR ; and these being tangents to the circles X, Z, O is a point on the radical axis of the circles X, Z.

i.e. O is a point on CN.

∴ AL, BM, CN all pass through O.

(ii) Secondly, suppose AL, BM are par^l.

Since the radical axis of any two circles is at right angles to the line of their centres,

AL is ⊥ to XY ; and BM is ⊥ to YZ.

Hence, since AL, BM are par^l, the str. lines XY, YZ, both of which have the pt. Y common, are in one and the same str. line.

∴ X, Y, Z lie on the same str. line.

Again, since the radical axis, CN, of the ⊙^s X, Z is ⊥ to XZ,

∴ AL, BM, CN are all parallel because they are perp^s to the same str. line.

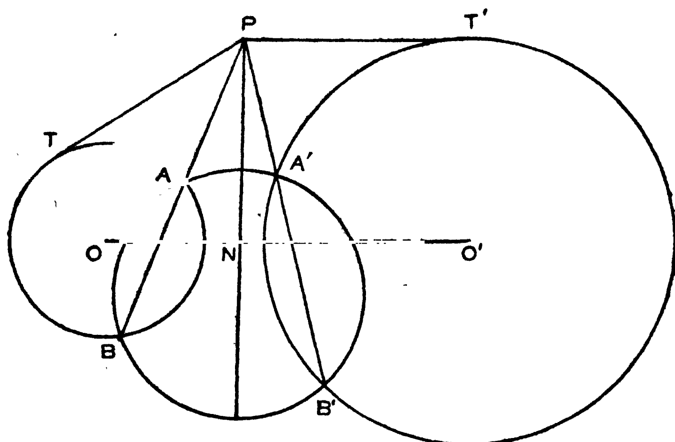
Obs. If the circles be such that only one of them intersects the other two, which are themselves non-intersecting, the truth of the above theorem may be established in a similar manner.

But if the circles intersect *each other*, two by two, the above theorem reduces to one which proves that the common chords of three circles, which intersect each other in pairs, are concurrent.

9. Definition : The point at which the radical axes of any three circles, taken in pairs, meet (if at all), is called their **radical centre.**

The pt. O in fig. 1 is the radical centre of the ⊙^s X, Y, Z.

4. *To construct the radical axis of two given circles, which do not intersect.*



Let TAB, T'A'B' be the given circles, which do not intersect; and let their centres be O and O' respectively.

It is required to construct their radical axis.

Construction. Draw any other ⊙ BAB' to cut the given ⊙s at A, B and A', B'. Join AB, A'B'; and produce them to meet at P.

Then the str. line through P at right angles to OO' is the reqd. radical axis.

Proof. Let PT and PT' be the tengents from P to the given ⊙s.

Then $PT^2 = PA.PB$; and $PT'^2 = PA'.PB'$.

But, BA and B'A' are chords of the ⊙BAB' and are produced to meet at P; ∴ PA.PB = PA'.PB'.

∴ $PT^2 = PT'^2$

∴ PT = PT',

i.e. P is a pt. on the radical axis.

10. **Definition:** If there be a system of circles such that the radical axis of *any two* of them is the same for any other pair, the circles are said to be **co-axial.**

MISCELLANEOUS EXERCISES

1. Show that the middle points of the chords of a circle which pass through a given point all lie on the circumference of another circle.

2. If a quadrilateral be circumscribed about a given circle, prove that the sum of any pair of opposite sides is equal to the sum of the other pair. Hence prove that any parallelogram circumscribed about a circle is a rhombus.

3. Prove that the triangle formed by joining the three points in which the in-circle of a triangle meets the sides, is acute-angled.

4. The diagonals AC, BD of a parallelogram ABCD intersect at O. Show that the circum-circles of the triangles AOB, COD touch each other at O.

5. Show that the greatest quadrilateral that can be inscribed in a circle is a square.

6. On the sides BC, CA, AB of a given triangle ABC, any three points D, E, F respectively are taken. Prove that the circum-circles of the triangles EAF, FBD, DCE meet in a point.

7. If the in-circle of a triangle ABC touches the sides BC, CA, AB at D, E, F respectively, prove that BC+AF=CA+BD=AB+CE=the semi-perimeter of the triangle ABC.

8. With the vertices A, B, C of a triangle as centres, construct three circles each of which shall touch the other two.

9. Prove that the sum of any three alternate angles of a hexagon inscribed in a circle is equal to four right angles.

10. AB is the common chord of two circles, and P is any point on the circumference of one of them so that the straight lines PAQ, PBR meet the circumference of the other in Q and R. Prove that the arc QR is of the same magnitude for different positions of P.

11. The opposite sides of a cyclic quadrilateral are produced to meet in P and Q, and about the triangles so formed without the quadrilateral circles are described, which meet in R. Prove that the points P, R, Q are collinear.

12. ABC is a triangle; BCA′, CAB′, ABC′ are equilateral triangles so that the points (A, A′), (B, B′) (C, C′) are on opposite sides of BC, CA, AB respectively. Prove that the circum-circles of the three equilateral triangles meet at a point which is also the point of concurrence of the lines AA′, BB′, CC′.

13. If an equilateral triangle be inscribed in a circle, and the adjacent arcs cut off by two of its sides be bisected, prove that the line joining the points of bisection is trisected by the sides.

14. P is any point on the circum-circle of a given triangle ABC; PA′, PB′, PC′ are chords of the circle perpendicular respectively to BC, CA, AB. Show that the triangles ABC, A′B′C are congruent.

15. If S be the circum-centre of a triangle ABC, and if the perpendicular from S on BC meets the circum-circle in K and L, L being on the same side of BC as A, prove that AK and AL bisect the interior and exterior angles at A. Also, if I be the in-centre, prove that K is the circum-centre of the triangle BIC.

16. ABC is a triangle; C is the point of intersection of the perpendiculars AD, BE, CF upon the sides of the triangle; G, H, K are the middle points of the sides, and L, M, N the middle points of the lines OA, OB, OC. Prove that each of the angles LHG, LKG, is a right angle; and hence prove that the circle passing through the points, G, H, K also passes through the six points (L, D), (M, E), (N, F).

17. Given one angle, of a triangle, the side opposite to it, and the sum of the other two sides, construct the triangle.

18. ABC is any triangle, inscribed in a circle, and AP, BQ are chords of the circle parallel to BC, CA respectively. Prove that PQ is parallel to the tangent at C.

19. AB is the diameter of a semi-circle, D and E are any two points on its circumference. AE, BD intersect at M; and AD, BE produced intersect at L. Prove that LM produced cuts AB at right angles.

20. BC is a given arc of a circle whose centre is O; A is any point in BC. AD, AE are drawn perpendiculars to OB, OC. Prove that the line DE is of constant length.

21. Prove that an equiangular polygon inscribed in a circle has its alternate sides equal. Hence show that an equiangular pentagon inscribed in a circle is also equilateral.

22. If any number of triangles be on the same base, on the same side of it, and have equal vertical angles, prove that the bisectors of the vertical angles are concurrent.

23. If two circles intersect and if one of them passes through the centre of other, prove that the tangents to the latter at the points of intersection will meet on the former.

24. If P be any point on the circle circumscribed about a given equilateral triangle ABC, prove that one of the lines PA, PB, PC is equal to the sum of the other two.

25. O is the circum-centre of a triangle ABC, and D, E, F are the feet of the perpendiculars from A, B, C on the opposite sides. Show that OA, OB, OC are respectively perpendiculars to EF, FD, DE.

26. Right-angled triangles are described on the same hypotenuse. Show that the locus of the centres of the in-circles is a quarter of the circumference of a circle of which the common hypotenuse is a chord.

27. Construct a triangle, having given one side and the radii of the in-circle and circum-circle.

28. ABCDE is a regular pentagon inscribed in a circle, and AC, BD intersect at O. Prove that AO = DO, and that $BC^2 = AC.CO$.

29. ACDB is a semi-circle. AB being the diameter, and the two chords AD, BC intersect at E. If a circle be circumscribed about the triangle CDE, prove that it will cut the former at right angles.

30. If I be the in-centre, and I_1, I_2, I_3, the ex-centres of a given triangle, prove that I is the ortho-centre of the triangle I_1, I_2, I_3. Hence show that the circum-circles of the four triangles II_2I_3, II_3I_1, II_1I_2 and $I_1I_2I_3$ are equal to one another.

31. The diagonals of a given quadrilateral ABCD intersect at O. Show that the centres of the circles circumscribed about the triangles OAB, OBC, OCD, ODA are at the angular points of a parallelogram.

32. If a line BC of constant length have its extremities on two fixed str. lines AX, AY, prove that the circum-radius of the $\triangle$ABC is of the same length for all positions of BC. Hence prove that, as BC changes its position, the circum-centre of the $\triangle$ABC moves on the $\bigcirc^{ce}$ of a $\odot$.

33. In the preceding example, show that the distance of the circum-centre from the line BC is the same for all positions of BC. Hence prove that, as BC changes its position, the ortho-centre of the triangle ABC moves on the circumference of a circle.

34. ABC is an equilateral triangle inscribed in a circle whose centre is O, and BO is produced to meet the circumference in D. Prove that the arc AD is one-sixth of the whole circumference. Hence prove that AD = AO, and that $AB^2 = 3OA^2$.

35. PM, PN are tangents to a circle, of which the centre is O, and MN the chord of contact. PEF is any straight line cutting the circle at E, F. OS is drawn perpendicular to EF and produced to meet MN

produced in Q. Prove that $OS.OQ = (\text{radius})^2$, and hence prove that QE, QF are tangents to the circle.

36. Through a given point without a given circle draw a straight line so that the part intercepted by the circumference may be equal to a given straight line not greater than the diameter.

37. The internal and external bisectors of the angle A of a triangle meet the base BC in E, E′ and the circum-circle in D and D′. Prove that D is the ortho-centre of the triangle EE′D′.

38. From S, the circum-centre of the triangle ABC, perpendiculars SA′, SB′, SC′ are drawn to the sides, and these perpendiculars are produced to P, Q, R respectively so that $SA' = A'P$, $SB' = B'Q$ and $SC' = C'R$. Prove that S is the ortho-centre of the triangle PQR, and hence show that $\triangle^s$ ABC, PQR have the same nine-point circle.

39. Construct a triangle, having given the ortho-centre, the circum-centre and one angular point.

40. AB and AC are two given str. lines in which B and C are two given pts. BD is drawn $\perp$ to AC, and DE $\perp$ to AB; in like manner CF is drawn $\perp$ to AB, and FG to AC. Prove that EG is $\parallel$ to BC.

41. If the centres of two circles which touch each other externally be fixed, prove that the external common tangents of the two circles will also touch the circle of which the straight line joining the fixed centres is the diameter.

42. AB, AC are the equal sides of an isosceles triangle ABC. P is a point on AB, and Q a point on AC produced such that $BP = CQ$. Prove that, for all positions of PQ, the circum-circle of the triangle APQ passes through a fixed point on the bisector of the angle BAC.

43. ABC is a triangle, and BE, CF are the perpendiculars from B and C upon the opposite sides. If K be the mid-pt. of BC, prove that $KF = KE$, and that each of the $\angle^s$ KFE, KEF is equal to the $\angle$A.

44. If two circles touch each other internally, prove that any chord of the greater circle which touches the smaller is divided at the point of contact into segments which subtend equal angles at the point of contact of the two circles.

45. PA, PB, PC are any three chords of a circle. Prove that the circles described on these chords as diameters will intersect again in three points which are collinear.

46. A series of circles touch a fixed straight line at a fixed point. Show that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

47. AD, BE, CF are the perpendiculars of the triangle ABC. Prove that the feet of the ⊥ˢ from D on AB, AC, BE, CF are collinear.

48. O is the ortho-centre of a △ABC, and P, Q, R the circum-centres of the △ˢ BOC, COA, AOB respectively. Prove that O is the circum-centre of the △PQR. Prove also that each of the quadls. OPCQ, and OQAR ORBP is a rhombus, and that the △ˢ ABC, POR are congruent.

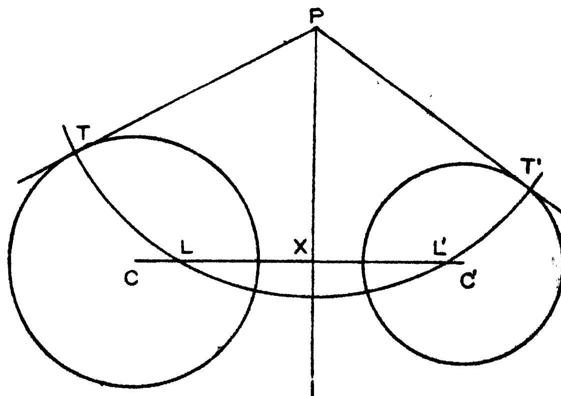
49. Having given the base and the vertical angle of a triangle, prove that the nine-point circle touches a fixed circle whose radius is equal to that of the circum-circle.

50. A man standing on the deck of a ship in the middle of a calm ocean looks around with a telescope. Show that the distance from him of the farthest object he can see on the surface of the ocean is very nearly $\sqrt{\frac{1}{8}}$ *mile*, where 1 *inch* is the height of the observe's eye above the water, assuming the radius of the earth to be 4000 miles.

51. If from a point P on the radical axis of two given circles a tangent PT be drawn to one of them, prove that the circle whose centre is P and radius is PT cuts the two given circles orthogonally.

52. Any circle which cuts two given non-intersecting circles orthogonally, pass through two fixed points. [a ⊙ whose centre is C and radius is r is sometimes denoted by (C, r).].

[Let P be the centre a ⊙ cutting the given ⊙ˢ (C, r), (C′, r')



orthogonally ; and let it cut CC′, the line of centres at L, L′.

Since PT, PT′ touch the given ⊙ˢ and PT = PT′, ∴ P is on the radical axis PX. Since, CT touches the ⊙ TLL′, ∴ CL.CL′ = CT² = r^2. Now X bisects LL′,

$$\therefore \quad XC^2 - XL^2 = CL.CL' = r^2$$
$$\therefore \quad XL^2 = XC^2 - r^2.$$

∴ L, L′ are fixed points.

53. In the diagram of Ex. 52, show that $C'L.\ C'L' = r'^2$.

54. If two circles touch, show that their radical axis is the tangent at their point of contact.

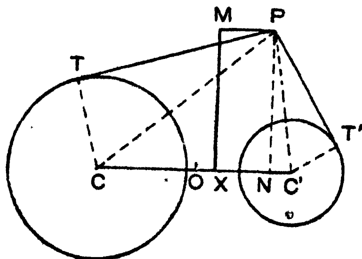
55. Show that the radical axis of two circles bisects their common tangents.

56. Find a point, by geometric construction so that the tangents drawn from it to three given non-intersecting circles are equal. What is the case of failure ?

57. *If* PT, PT′ *are tangents from any point* P *to two circles* (C, r), (C, r') *and* PM *is the perpendicular from* P *to the radical axis of the circles, then*

$$PT^2 - PT'^2 = 2CC'.\ PM,$$

it being assumed that PT > PT′.



Proof. Let the radical axis meet CC′ at X. Draw PN perpendicular to CC′.

Because ∠ˢ PTC, PNC are right angles,

$$\therefore PT^2 + CT^2 = CP^2$$
$$= CN^2 + PN^2,$$
$$\therefore PT^2 + r^2 = CN^2 + PN^2.$$

Similarly,

$$PT'^2 + r'^2 = C'N^2 + PN^2\ ;$$
$$\therefore PT^2 - PT'^2 + r^2 + r'^2 = CN^2 - C'N^2.$$

Also, since MX is the radical axis,

$$r^2 - r'^2 = CX^2 = C'X^2 = 2OX.CC'\ ;$$

and $$CN^2 - C'N^2 = (CN - C'N)(CN + C'N) = 2ON,\ CC'\ ;$$

$$\therefore PT^2 - PT'^2 + 2OX.CC' = 2ON.CC'\ ;$$
$$\therefore PT^2 - PT'^2 = 2XN.CC' = 2CC',\ PM.$$

58. As a particular case of (57) let P be any point on the circle (C′, r'). In this case PT′ vanishes, and we have

$$PT^2 = 2CC'\ PM.$$

UNIVERSITY MATRICULATION PAPERS

CALCUTTA

COMPULSORY PAPER

1925

1. (i) Prove that the sum of any two sides of a triangle is greater than the third side.

(ii) Two sides of a triangle are 2 and 3; show that the third side is less than 5 but greater than 1.

2. (i) Prove that any pair of opposite angles of a cyclic quadrilateral are supplementary.

(ii) Show that the internal bisectors of the angles of any quadrilateral form a cyclic quadrilateral.

3. *Either*, (i) If two chords of a circle intersect outside the circle, show that the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

(ii) Deduce from this that the two tangents to a circle which can be drawn from an external point are equal.

1926

1. *Either*, (i) If the sides of a triangle are equal, prove that the angles opposite to them are also equal.

(ii) Prove that the angles at the base of an isosceles triangle are acute.

Or, (i) Prove that the diagonals of a parallelogram bisect each other.

(ii) If the diagonal AC of a parallelogram ABCD bisects the angle A, show that it bisects the angle C and the parallelogram is a rhombus.

2. *Either*, (i) Prove that the sum of the angles of a triangle is two right angles.

(ii) The sum of the base angles of a triangle is 108° and their difference is 12°. Find all the angles of the triangle.

Or, (i) Prove that equal chords of a circle are equidistant from the centre.

(ii) AB and AC are two equal chords of a circle ; show that the bisector of the angle BAC passes through the centre.

3. *Either*, (i) Prove that the two tangents drawn to a circle from an external point are equal and subtend equal angles at the centre of the circle.

(ii) Show that the centre of any circle touching two intersecting straight lines lies on the bisectors of the angles between them.

Or, Bisect a given arc of a circle (statement, as well as justification, of construction is required.)

1927

1. *Either*, (i) Prove that any two sides of a triangle are together greater than the third side.

(ii) Prove that the sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

Or, (i) Prove that the opposite sides and angles of a parallelogram are equal to one another.

(ii) One angle of a parallelogram is a right angle. Prove that it is a rectangle.

2. (i) Prove that the angle in a semi-circle is a right angle.

(ii) A circle is described on the hypotenuse of a right-angled triangle as diameter. Prove that the circle passes through the opposite angular point.

3. Bisect a given arc of a circle. [Only the traces of construction are required.]

1928

1. *Either*, (i) If one angle of a triangle be greater than another, prove that the side opposite to the greater angle shall be greater than the side opposite to the less.

(ii) Hence deduce that the hypotenuse is the greatest side in a triangle.

Or, (i) Prove that the three interior angles of a triangle are together equal to two right angles.

(ii) If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.

2. (i) In equal circles, prove that the arcs which subtend equal angles whether at the centres or circumferences shall be equal.

(ii) Two equal circles intersect at A and B ; and through A any straight line PAQ is drawn terminated by the circumferences. Show that BP = BQ.

3. Prove that the angle at the centre of a circle is double the angle at the circumference standing on the same arc.

1929

1. *Either*, (i) Prove that if two straight lines intersect, the vertically opposite angles are equal.

(ii) Two straight lines AB and CD intersect at E. If the bisector of the angle AEC be produced, prove that it will bisect the angle BED.

Or, (i) Prove that two triangles are equal in every respect, if two angles and the adjacent side of one triangle are respectively equal to two angles and the adjacent side of the other.

(ii) The triangle ABC has the angles at B and C equal. Show that the bisectors of these equal angles terminated by the opposite sides are equal.

2. (i) Prave that if two tangents are drawn to a circle from an external point they are equal.

(ii) If the circumference of a circle is divided into three equal arcs, the tangents drawn to the circle at the points of section form an equilateral triangle.

3. Draw a tangent to a given circle from an external point. (Traces of construction must be given, but no justification is required.)

1930

1. *Either*, (i) Prove that the three angles of a triangle are together equal to two right angles.

(ii) Find in *degrees* each angle of a polygon of five sides. Give reasons for your answer.

Or, (i) Prove that the area of a triangle is half the area of a parallelogram on the same base and of the same altitude.

(ii) ABCD is any parallelogram and O is any point within it. Show that the sum of the areas of the triangles AOB and COD is equal to half the area of the parallelogram.

2. *Either*, (i) Establish geometrically the algebraically formula $a^2 - b^2 = (a+b)(a-b)$.

(ii) In a triangle ABC, AD is the perpendicular drawn to the base BC and O is the middle point of BC. Prove that the difference $AB^2 \sim AC^2 = 2BC.OD$.

Or, (i) Prove that the tangent at any point of a circle is at right angles to the radius drawn through the point.

(ii) The radius of a given circle is 1·5 inches. Prove that all points from which the tangents drawn to the circle are of constant length 2 inches lie on a circle. Draw a diagram as accurately as you can.

3. Construct a triangle whose base will be 6 centimetres and the other two sides 3 and 5 centimetres respectively. Measure as accurately as possible the altitude of the triangle.

[*Traces and statement of construction are required.*]

1931

1. *Either*, (i) If two angles of one triangle are respectively equal to two angles of another, and the side adjacent to the angles in one equal to the side adjacent to the equal angles in the other, prove that the two triangles are equal in all respects.

(ii) A diagonal of a parallelogram is bisected, and through the point of bisection a straight line is drawn to be terminated by one pair of opposite sides. Show that the straight line is bisected at the point.

Or, (i) Prove that any two sides of a triangle are together greater than the third side.

(ii) Prove that the difference of any two sides of a triangle is less than the third side.

2. *Either*, (i) Prove the geometrical proposition corresponding to the algebraical formula $(a+b)^2 = a^2 + b^2 + 2ab$.

(ii) Prove that the square on a straight line is equal to four times the square on half the line.

Or, (i) Draw two tangents to a circle from an external point.

(ii) A quadrilateral is described touching a circle. Prove that the sum of any pair of opposite sides is equal to the sum of the other pair.

3. Construct a triangle, given the base, one side, and the area.

1932

1. *Either*, (i) If one side of a triangle is produced prove that the exterior angle is greater than either of the interior opposite angles.

(ii) Show that it is impossible to draw three equal straight lines from a given point to a given straight line.

Or, (i) Prove that, if a straight line cuts two parallel straight lines, the corresponding angles are equal.

(ii) Prove that, if the three sides of one triangle are parallel to the three sides of another triangle, the corresponding angles are equal.

2. *Either*, (i) If a straight line drawn through the centre of a circle bisects a chord which does not pass through the centre, prove that it cuts the chord at right angles.

(ii) Show how to construct a circle of given radius to pass through two given points. When is this construction impossible ?

Or, (i) Prove that the tangent at any point of a circle and the radius through the point are perpendicular to one another.

(ii) Show how to draw a tangent to a given circle parallel to a given straight line. How many such tangents are possible ?

3. (i) Construct a square on a given finite straight line. (Give only the traces of *all* your constructions, using a hard pencil, a straight ruler, and a pencil compass only).

(ii) Divide the area of a given square into parts from which two equal squares can be made up.

1933

1. *Either*, (i) Show that in a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

(ii) Prove that in an equilateral triangle four times the square on the perpendicular drawn from a vertex on the opposite side is equal to three times the square on any side.

Or, (i) Show that in an obtuse-angled triangle the square on the side subtending the obtuse angle is greater than the sum of the squares on the other two sides by twice the rectangle contained by one of those sides and the projection of the other side upon it.

(ii) Prove that a triangle whose sides are 2, 3, and 4 inches is an obtuse-angled triangle.

2. *Either*, (i) Show that equal chords of a circle are equidistant from the centre.

(ii) Find the locus of the mid-points of chords of constant length in a circle.

Or, (i) Show that there is only one circle which passes through three given points not in a straight line.

(ii) Prove that two different circles cannot cut each other at more than two points.

3. (i) Describe a parallelogram equal in area to a given triangle and having one of its angles equal to a given angle. (Traces only are required.)

(ii) Construct a rhombus equal in area to a given rectangle and having a side equal to a side of the rectangle. (Traces only are required.)

DACCA

1932

1. *Either*, (i) Any two sides of a triangle are together greater than the third side.

(ii) Show that any two sides of a triangle are together greater than twice the median bisecting the third side.

Or, (i) The three angles of a triangle are together equal to two right angles.

(ii) ABC is an isosceles triangle of which the side AB is equal to the side AC. BA is produced to D so that AD is equal to AB. Prove that BCD is a right angle.

2. *Either*, (i) In any triangle, the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing that angle diminished by twice the rectangle contained by one of those sides and the projection of the other side upon it.

(ii) The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

Or, (i) The angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.

(ii) A tangent is drawn parallel to a chord ; show that the intercepted arc is bisected at the point of contact.

3. Take a straight line AB 2·5″ long. At A make an angle BAC equal to 60°, and at B draw BC perpendicular to BA, meeting AC in C. Bisect AC at D. Measure BD.

[*No proof is required but traces of construction must be left on the paper. Only ruler and compass are to be used.*]

1933

1. *Either*, (i) If two angles of a triangle ars equal, the sides opposite to these angles are also equal.

(ii) Prove that the hypotenuse of a right-angled triangle is double the median which bisects the hypotenuse.

Or, (i) Equal triangles standing on the same base and on the same side of it are between the same parallels.

(ii) Prove that the line joining the middle points of any two sides of a triangle is parallel to the third side.

2. (i) Enunciate and prove the geometrical theorem corresponding to the algebraical identity

$$a^2-b^2=(a+b)(a-b).$$

(ii) If a straight line is bisected and also divided into two unequal segments, the rectangle contained by these segments is equal to the difference of the squares on half the line and on the line between the points of section.

3. *Either*, (i) The tangent at any point of a circle is perpendicular to the radius through the point.

(ii) Find the locus of a point which moves in such a manner that tangents from it to a fixed circle are of the same constant length.

Or, Bisect a quadrilateral by a straight line drawn through an angular point.

[*Traces of construction as also proof are necessary.*]

ALLAHABAD

1931

1. (i) If there are three or more parallel straight lines, and the intercepts made by them on any transversal are equal, then the corresponding intercepts on any other transversal are also equal.

(ii) In a triangle ABC, if a set of lines Pp, Qq, Rr......drawn parallel to the base, divide one side AB into equal parts, they also divide the other side AC into equal parts.

2. (i) Prove that if opposite angles of a quadrilateral are supplementary, its vertices are concyclic.

(ii) ABC is an isosceles triangle, and XY is drawn parallel to the base BC cutting the sides in X and Y, show that the four points B, C, X, Y lie on a circle.

3. (i) Show that in any triangle the sum of the squares on two sides is equal to twice the squares on half the base together with twice the square on the median which bisects the base.

(ii) In any triangle the difference of the squares on two sides is equal to twice the rectangle contained by the base and the intercept between the middle point of the base and the foot of the perpendicular drawn from the vertical angle to the base.

4. (i) Prove that the distance of each vertex of a triangle from the orthocentre is double of the perpendicular drawn from the centre of the circle circumscribed round the triangle to the opposite side.

Or, (ii) Construct a triangle, having given a vertex. the orthocentre, and the centre of the circle circumscribed round the triangle.

1932

1. Prove that two tangents can be drawn to a circle from an external point. Further prove that these two tangents are equal and subtend equal angles at the centre.

2. (i) In a right-angled triangle, prove that the square on the hypotenuse is equal to the sum of the squares on the other two sides.

(ii) ABC is a right-angled triangle, A being the right angle; and the sides AB, AC are intersected by a straight line PQ at P and Q respectively, and BQ, PC are joined. Prove that $BQ^2+PC^2=BC^2+PQ^2$.

3. (i) If two chords of a circle are equal they are equidistant from the centre ; conversely, two chords which are equidistant from the centre are equal.

(ii) If a chord of a circle subtends a constant angle at the centre, show that the locus of its middle point is a concentric circle.

4. Prove the concurrence of—

Either, (i) the perpendiculars drawn from the vertices of a triangle to the opposite sides ;

Or, (ii) the perpendiculars drawn to the sides of a triangle from their middle points.

PATNA

1932

1. *Either*, (i) Prove that the sum of the angles of a triangle is equal to two right angles.

(ii) Show that the angle contained by the bisectors of the adjacent angles A and D of the quadrilateral ABCD is half the sum of the remaining angles.

Or, (i) If two triangles have three sides of the one equal to three sides of the other, each to each, then the triangles are equal in all respects

(ii) ABC is an isosceles triangle in which AB=AC ; if the bisectors of the angles B and C meet in O, show that AO bisects the angle A.

2. *Either*, (i) Enunciate and prove the theorem which corresponds to the formula

$$(a+b)(a-b)=a^2-b^2.$$

(ii) The straight line AB is bisected at C and produced to D. Prove that $AD^2-DB^2=4AC.CD$.

Or, (i) Prove that the tangents drawn to a circle from an external point are equal and subtend equal angles at the centre.

(ii) Find the locus of points from which pairs of equal tangents can be drawn to two intersecting circles.

3. Draw a parallelogram ABCD in which the two adjacent sides AB=5·5 cm., AB=7·7 cm., and $\angle A=60°$. From the angles A and B draw perpendiculars to BC and AC to meet them (produced if necessary) at E and F respectively. Bisect AB at G. Measure the lengths EG and FG.

[*Traces of construction only are required.*]

1933

1. A straight line intersects two parallel straight lines. Describe experiments to establish any relation that exists between (1) the alternate angles, (2) the corresponding angles, (3) the interior angles on the same side of the intersecting lines.

State the general truths you may deduce from these experiments.

2. Construct a triangle whose base is 6·5 cm. and one of the angles at the base is 60° and the sum of the other two sides is 10 cm. Measure the lengths of these two sides separately.

[*Traces of constructions only are required.*]

3. (i) A chord is drawn from the point of contact of a tangent to a circle. Prove that the angle between the tangent and the chord is equal to the angle in the alternate segment of the circle.

(ii) Two circles touch internally at A. PQ, a chord of the outer, touches the inner at R. Prove that AR bisects the angle PAQ.

Or, (i) Prove that, in an obtuse-angled triangle, the square on the side opposite to the obtuse angle is greater than the sum of the squares on the other two sides by twice the rectangle contained by one of the two sides and the projection on it of the other.

(ii) D is a point in the base BC of an isosceles triangle ABC. Prove that $AB^2 = AD^2 + BD.CD$.

4. Explain what is meant by 'the locus of a point'.

Find the locus of a point which moves so that its distances from two fixed points are always equal.

BOMBAY

1932

1.—(*a*) Prove that if two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent. (Name the triangles LMN and FGH.)

(*b*) ABCD is a rhombus whose diagonals intersect at O. On AB draw a semi-circle AEB, remote from the side on which ABCD stands. If a chord AE equal to BO is placed in the semi-circle AEB, prove that BE is equal and parallel to AO.

2. (*a*) Prove that in any triangle the sum of the three angles is equal to two right angles. (Do not take it for granted without actually proving that the exterior angle is equal to the sum of the two interior and opposite angles.)

(*b*) ABC is a triangle in which AB is equal to AC. ∠B is bisected by BD cutting AC in D. BD is equal to AD. Prove that ∠A=36° and AD=BC.

3. (*a*) Construct a parallelogram ABCD such that its diagonal AC=4 cm. and side AB=4.5 cm., and its area is 12 sq. cm.

(*b*) Construct a triangle ABC such that BC=4 cm., ∠A=60° and AC+AB=6 cm. Measure AB and AC.

4. (*a*) Two equal triangles are on equal bases in the same straight line and on the same side of it. Prove that they stand between the same parallels.

(*b*) R is the mid-point of the side AB of a triangle ABC. P is any point on AC. BP is produced to a point S such that the triangle RPS is equal to the triangle RCP. Prove that CS is parallel to AB.

5. (*a*) Prove that in any triangle the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median bisecting the third side.

(*b*) P and Q are the mid-points of the diagonals of a quadrilateral. Prove that the sum of the squares on the four sides of the quadrilateral exceeds the sum of the squares on the diagonals by $4PQ^2$.

6. (*a*) If a straight line joining two points subtends equal angles at two other points on the same side of it, prove that the four points lie on a circle.

(*b*) AD, BE and CF are the altitudes of an acute-angled triangle ABC. Prove that ∠AFE=∠BFD and prove hence or otherwise that ∠DFE is the supplement of twice the angle ACB.

7. (*a*) If a straight line touch a circle and if from the point of contact a chord of the circle be drawn, prove that the angles between the tangent and the chord are equal to the angles in the alternate segments.

(*b*) Two circles touch each other internally at A. P is any point on the common tangent at A. The other tangent from P to the inner circle touches it at Q and cuts the outer circle in R and S. Prove that AQ bisects the angle RAS.

8. (*a*) Construct a triangle whose perimeter is 11 cm., and whose three sides are in the ratios of 2 : 3 : 4. State briefly the main steps of your construction.

(b) Describe a circle with a radius of 3 cm. and construct tangents to it from a point 5 cm. apart from the centre of the circle. Calculate the length of each tangent and measure the angle included between the tangents.

1933

1. (a) If two sides of a triangle are equal, prove that the angles opposite to these sides are equal. State the converse of the theorem.

(b) ABCD is a trapezium in which AB is parallel to DC and AD is equal to BC. Prove that the angle ADC is equal to the angle BCD.

2. (a) If two triangles have two sides of the one equal to two sides of the other, each to each, and also the contained angle of the one equal to the contained angle of the other, prove that the triangles are congruent. Name the triangles CDE and FGH.

(b) If the bisector of the vertical angle of a triangle bisects the base, prove that the triangle is isosceles.

3. (a) Prove that the locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.

(b) Construct a quadrilateral ABCD such that $\angle D=60°$, $\angle A=\angle C=90°$ and BA=BC=3 cm. Measure BD. State your construction.

4. (a) Illustrate geometrically $a^2-b^2=(a+b)(a-b)$.

(b) ABC is a triangle right-angled at B. BD is drawn perpendicular to AC meeting AC in D. BE is drawn to meet the hypotenuse in E so that $\angle EBC=ECB$. Prove that $BC^2-AB^2 =2DE.AC$.

5. (a) Prove that the perpendiculars of a triangle meet in a point. (Ortho-centre.)

(b) Prove that these perpendiculars bisect the angles of the triangle formed by joining the feet of these perpendiculars.

6. (a) Prove that the angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference. Hence deduce that angles in the same segment of a circle are equal.

(*b*) Inscribe in a circle of radius 3 cm. a triangle whose angles are 40°, 60° and 80°. Measure the side opposite to the angle of 60°. State your construction.

7. (*a*) If two circles touch, prove that the point of contact lies on the straight line through the centres of the circles.

(*b*) Two circles touch externally at C. Through C lines ACD and BCE are drawn making equal angles with the line joining the centres, so that A and B lie on one circle and E and D on the other circle. Prove that AD = BE and AB is parallel to ED.

8. Two circles of radii 3 cm. and 4 cm. have their centres P and Q respectively 5 cm. apart from each other, and intersect each other at C and D. If CD intersects PQ in X, prove without measuring that (i) $\angle$PCQ and $\angle$PDQ are right angles, (ii) XQ $= \frac{16}{5}$ cm., PX $= \frac{9}{5}$ cm., (ii) CD $= \frac{24}{5}$ cm., (iv) the length of the tangent to either circle from a point on CD distant $\frac{8}{5}$ cm. from C or D is $\frac{16}{5}$ cm.

MADRAS

1931

1. (i) Show that the sum of any two sides of a triangle is greater than the third side.

(ii) If D is the middle point of the side BC of the triangle ABC, show that AB + AC > 2AD.

2. (*i*) Prove that the diagonals of a parallelogram bisect each other.

(ii) Prove that any straight line drawn through the middle point of a diagonal of a parallelogram bisects the area of the parallelogram.

3. (i) Enunciate and prove the geometrical theorem corresponding to the algebraical identity $a^2 - b^2 = (a+b)(a-b)$.

(ii) AD is the perpendicular on the side BC of a triangle ABC and E is the middle point of BC. Show that $AB^2 \sim AC^2 = 2.BC.ED$.

4. The sides of a quadrilateral taken in order are 3·4, 1·5, 2·3, and 2·6 inches. The angle between the first two sides is 73°. Draw

the quadrilateral. Construct a square equal in area to the quadrilateral and measure its side.

5. (i) If the opposite angles of a quadrilateral are supplementary, prove that it can be inscribed in a circle.

(ii) ABCD is a rectangle in which AB=3″, BC=2″. E and F are points on BC, CD such that CE=1″ and CF=2″. Prove that ABEF is cyclic.

6. (i) AB and CD are two chords of a circle intersecting at O. Prove that the rectangles OA.OB and OC.OD are equal.

(ii) If two circles are concentric, show that the difference of the squares on the tangents from an external point to the two circles is equal to the square of the tangent from any point on the outer circle to the inner.

7. A circle of radius 2″ is described with the centre at the origin. Describe another circle of radius 1″ to touch the former externally and the x-axis.

PUNJAB

1932

PART I.

1. Draw a triangle whose sides are respectively 3·1, 4·2 and 5·3 inches. Describe the inscribed and circumscribed circles, and measure the distance between their centres.

2. The transverse common tangents of two circles of radii 1 inch and ½ inch respectively are at right angles to each other. Construct the circles, explaining clearly the steps of construction.

3. (*a*) If the sides of a convex polygon are produced in order, prove that the sum of the angles so formed is equal to four right angles.

(*b*) If each angle of a rectilineal figure is equal to seven-eighths of two right angles, find the number of its sides.

4. (*a*) Prove that, in any triangle, the sum of the squares on two sides is equal to twice the square on half the base together with twice the square on the median which bisects the base.

(*b*) Show that three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians of the triangle.

5. (*a*) Prove that the locus of a point which is equidistant from two intersecting straight lines consists of a pair of straight lines.

(*b*) A chord of a circle is of constant length, but its position changes. A point is taken on it at a fixed distance from one of its ends. Prove that this point always lies on a fixed circle.

PART II.

6. Inscribe an equilateral triangle in a circle of radius 2·6 cm. and construct a square equal in area to the triangle. Measure the side of the square.

7. (*a*) Prove that the opposite angles of a cyclic quadrilateral are supplementary.

(*b*) If a side DC of a cyclic quadrilateral ABCD be produced to T, show that the exterior angle BCT is equal to the opposite interior angle BAD. What does this proposition become in the limit when D coincides with C ?

8. (*a*) Prove that if two chords of a circle intersect inside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

(*b*) Show that if two chords of a circle bisect one another they must be diameters.

9. (*a*) If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, show that the triangles are equiangular.

(*b*) Hence, prove that if from a point O outside a circle two straight lines OAB and OT be drawn to meet the circle in A, B and T such that $OT^2 = OA.OB$, then OT will be a tangent to the circle.

10. (*a*) Two circles touch internally at A, and a chord APQ is drawn cutting them in P and Q. If the tangent at P meet the outer circle in H and K, prove that HQ=KQ.

(*b*) If L, M, N be the feet of the perpendiculars from the vertices A, B, C respectively of an acute-angled triangle ABC to the opposite sides, prove that AL bisect the angle MLN.

1933

PART I

1. The sides AB, BC of a triangle ABC are 2·8 and 2·1 inches respectively and the angle BAC is 30 degrees. Construct two *different* triangles with the given data. Measure the length of the perpendicular from B to the opposite side and verify.

2. (*a*) Prove that in a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.

(*b*) If ABC be an equilateral triangle, and AL the perpendicular from A to BC, show that $AL^2 = 3BL^2$.

3. (*a*) Show that if two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

(*b*) Hence, prove that the sum of two sides of a triangle is greater than the third.

(*c*) If AD be the median, bisecting the base BC of a triangle ABC, show that the sum of the sides AB and AC is greater than twice AD.

4. (*a*) Prove that the perpendiculars from the vertices of a triangle to the opposite sides meet in a point (called the ortho-centre).

Also prove that the distance of each vertex from the ortho-centre is twice the perpendicular distance of the circum-centre from the side opposite to that vertex.

Or, (*b*) Prove that an angle in a semi-circle is a right angle.

If AL, BM, CN be the perpendiculars from the vertices of a triangle ABC to the opposite sides meeting in O, prove that

$$OA.OL = OB.OM = OC.ON.$$

[*Hint* : Prove ABLM concyclic, etc.]

PART II.

5. (*a*) Draw two circles of radii 3 cm. and 4·5 cm., whose centres are 5·7 cm. apart. Draw their common chord and a common tangent. Measure the length of the common chord.

(*b*) Bisect a given triangle by drawing a straight line parallel to the base.

6. (*a*) If two triangles are equiangular, prove that the side opposite to the corresponding angles are proportional.

(*b*) ABCD is a parallelogram. Through C a straight line MN is drawn outside the parallelogram, and AL, BM, DN are drawn perpendiculars to MN. Show that BM+DN=AL.

[*Hint*: Through B draw a straight line parallel to MN, etc.]

7. (*a*) If a straight line touch a circle, and from the point of contact a chord be drawn, prove that the angles which this chord makes with the tangent are equal to the angles in the alternate segment.

(*b*) Two circles touch internally at P, and a straight line ABCD meets the outer circle in A and D and the inner circle in B and C. Prove that ∠APB=∠CPD.

8. (*a*) Show that the angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

If any number of angles be inscribed in the same segment of a circle, prove that their bisectors pass through a common point.

Or, (*b*) Prove that the interior angles of a triangle are together equal to two right angles.

If two straight lines are perpendicular to two other straight lines, each to each, show that the acute angle between the first pair is equal to the acute angle between the second pair.

ANSWERS

TO

NUMERICAL EXERCISES

BOOK I

EXERCISE 2. [Pages 19—20.]

1. (i) 150° ; (ii) 135° ; (iii) 113° ; (iv) 60° ; (v) $\frac{4}{3}$ of a rt. ∠ ; (vi) 64°37′ ; (vii) 154°9′25″ ; (viii) 44°28′13″ ; (ix) $\frac{2}{3}$ of a str. ∠. **2.** (i) 60° ; (ii) 45° ; (iii) 30° ; (iv) $\frac{2}{3}$ of a rt. ∠ ; (v) $\frac{2}{5}$ of a rt. ∠ ; (vi) $\frac{4}{5}$ of a rt. ∠ ; (vii) 54°10° ; (viii) 28°27′45″ ; (ix) 8°42′13″. **3.** (i) 135° ; (ii) 120° ; (iii) 94°25′. **4.** (i) 60° ; (ii) 73° ; (iii) 45°. **5.** (i) 30° ; (ii) 45° ; (iii) 60°. **6.** 180° ; 90°. **10.** 60°.

EXERCISE 8. [Page 21.]

1. ∠AOC=30° ; ∠BOC=∠DOA=150°.

2. ∠AOC=60°=∠BOD ; ∠DOA=120°.

3. ∠AOC=∠BOD=45° ; ∠COB=∠DOA=135°.

4. ∠BOC=∠BOD=90°.

EXERCISE 11. [Page 48.]

5. ∠BQP=60°=∠AQR ; ∠BQR=∠AQP=120° ; ∠CRQ=120° ; ∠DRQ=60° ; ∠CRS=60° ; ∠DRS=120° ; ∠ESR=120° ; ∠FSR=60° ; ∠EST=60° ; ∠FST=120°.

EXERCISE 13. [Page 58.]

2. 45°, 45°, 90°. **4.** 90°. **6.** yes.

EXERCISE 14. [Pages 60—63.]

1. 54° each. **2.** 36°. **3.** 60°. **4.** 120°. **5.** 20°, 40°, 120°. **6.** (i) 95° ; (ii) 77° ; (iii) 49°. **9.** 51°, 74°, 55°.

EXERCISE 15. [Page 67.]

1. (i) 6 rt. ∠s ; (ii) 14 rt. ∠s ; (iii) 12 rt. ∠s ; (iv) 18 rt. ∠s ; (v) 26 rt. ∠s. **2.** (i) 3 ; (ii) 4 ; (iii) 5 ; (iv) 6 ; (v) 12.
3. (i) 40° ; (ii) 30° ; (iii) 24° ; (iv) $(25\frac{4}{7})°$. **4.** (i) 18 ; (ii) 10 ; (iii) 9 ; (iv) 4 ; (v) 3. **5.** (i) 3 ; (ii) 4 ; (iii) 5 ; (iv) 10 ; (v) 12. **6.** The least ∠ = 60° ; the other four ∠s are each = 120°. **7.** 3. **8.** 4.

EXERCISE 18. [Pages 89—90.]

1. (i) 60°, 120°, 120° ; (ii) 135°, 45°, 45° ; (iii) 150°, 30°, 30°.
2. ∠BAD = 75° ; ∠BCD = 75° ; ∠ADC = ∠ABC = 105°.

MISCELLANEOUS EXERCISES [Pages 113—116.]

17. 80°, 60°, 40°. **18.** 60°, 30°, 90°. **19.** 40. **20.** 14. **21.** 21.

BOOK II

EXERCISE 1. [Page 121.]

3. 10 sq. cms.

EXERCISE 2. [Pages 129—131.]

1. 42 sq. cms. **2.** 12′. **3.** 5″. **4.** 22·5 sq. ft. **5.** 5 cms.
6. 135 sq. ft. *i.e.*, 15 sq. yds. **7.** (i) 70 sq. cms. ; (ii) 68 sq. ft. ; (iii) 10 sq. yds. **8.** (i) 70 sq. cms. ; (ii) 115 sq. ft. ; (iii) 702 sq. ft. **9.** (i) 25 sq. inches ; (ii) 49 sq. cms. ; (iii) 9 sq. yds. **10.** (i) 24·5 sq. ft. ; (ii) 72 sq. cms. ; (iii) 84·5 sq. yds. **31.** $4\frac{8}{13}$ ft.
32. (i) 80 sq. ft. ; (ii) 96 sq. ft. ; (iii) 45 sq. cms.

EXERCISE 4. [Pages 148—150.]

1. (i) 5 ; (ii) 13 ; (iii) 17 ; (iv) 26 ; (v) 25 ; (vi) $\sqrt{410}$ *i.e.*, 20·248, nearly. **2.** (i) 24
(iii) 24 ; (iv) 7 ; (v) 8·3, nearly. **4.** 13 r

5. 35 ft. **6.** 25 ft. **7.** 60 ft. **8.** One foot.

9. (i) 5·657 feet nearly. (ii) $32\sqrt{2}$ cms. (iii) $15\sqrt{2}$ yds.; (iv) $a\sqrt{2}$ ft. **10.** (i) 13 ft.; (ii) 17 cms. (iii) $\sqrt{a^2+b^2}$ cms. **11.** 240 sq. ft. **12.** 37·5 cm.

13. 5·1 cm., each. **14.** 7·2 ft.; 15·12 sq. ft.

15. 12·99 ft., nearly; 97·425 sq. ft.

MISCELLANEOUS EXERCISES. [Pages 155—157.]

30. (i) $\sqrt{a^2+b^2}$; (ii) $\frac{1}{2}ab$; (iii) $ab\sqrt{a^2+b^2}$; (iv) $\frac{1}{2}\sqrt{a^2+b^2}$.

39. (i) 84 sq. cm.; (ii) 30 sq. ft.; (iii) 48·59 sq. cm.; (iv) $\frac{135}{4}\sqrt{39}$ sq. ft. **40.** 84 sq. ft.; 6·72 ft. **41.** 324 sq. ft.

BOOK III

EXERCISE 2. [Page 167.]

4. 2·5 inches.

EXERCISE 3. [Page 170.]

2. 8 inches.

EXERCISE 11. [Page 214.]

2. Two solutions. **6.** Four solutions. **10.** Four solutions.

EXERCISE 12. [Page 219.]

3. The reqd. distances are each = 4·87 cms. nearly. **10.** Two solutions. **18.** There is only one transverse common tangent (*viz.* the one at their pt. of contact because the two ⊙s in this case touch externally.)

EXERCISE 14. [Page 227.]

2. 2·5″. **4.** 1·73″, nearly.

EXEECISE 15A. [Page 231.]

Circumscribed regular hexagon = $\frac{4}{3}$ × inscribed regular hexagon.

BOOK IV

EXERCISE 1. [Page 255.]

2. 3750 sq. ft., each.

EXERCISE 4. [Pages 273—274.]

1. 4″. **2.** 4″. **3.** 5″.

MISCELLANEOUS EXERCISES. [Pages 274—276.]

10. When the two parts are equal. **14.** $\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c denote the sides of the Δ, s its semi-perimeter (a being the base). **15.** 84 sq. inches. **16.** 12·5″. **17.** 2″. **25.** 14·5″.

EXERCISE 5. [Page 284.]

4. (i) 7, 6 ; 6, 7. (ii) 8, 5 ; 5, 8. **5.** (i) $x=10, y=4$; ($x=-4, y=-10$) ; (ii) $x=12$, $y=6$; ($x=-6$, $y=-12$).
6. (i) 2·64, nearly ; (ii) 2·82, nearly ; (iii) 3. **7.** (i) 1, 2 ; (ii) 1·236, or (−3·236), nearly.